SEMIGROUP PRESENTATIONS

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Contents

To
my wife and children

Declaration

I, Mohammed Ali Faya Ibrahim, hereby certify that this thesis has been composed by myself, that it is a record of my own work, and that it has not been accepted in any previous application for any degree.

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Abstract

In this thesis we consider the following two fundamental problems for semigroup presentations :

1. Given a semigroup find a presentation defining it.

2. Given a presentation describe the semigroup defined by it.

We also establish other related results.

After an introduction in Chapter 1, we consider the first problem in Chapter 2, and establish a presentation for the commutative semigroup of integers $\mathbf{Z}_{p^t}$. Dually, in Chapter 3 we consider the second problem and study presentations of semigroups related to the *direct product of cyclic groups.* In Chapter 4 we study presentations of semigroups related to *dihedral groups* and establish their *$\mathcal{D}$-classes structure* in Chapter 5. In Chapter 6 we establish some results related to the Schützenberger group which were suggested by our studies of the semigroup presentations in Chapters 3 and 4. Finally, in Chapter 7 we define and study new classes of semigroups which we call R, L-semi-commutative and semi-commutative semigroups and they were also suggested by our studies of the semigroup presentations in Chapters 3 and 4.

Notation

Number theory

$m \pmod{n}$	m modulo n
$n \mid m$	n divides m
(n, m)	greatest common divisor of n and m
$[n, m]$	least common multiple of n and m
$ord_m\ a$	the order of a modulo m
$\phi(n)$	ϕ-function.

Semigroups

$\mid$	restriction of a mapping to a set
A^*	the set of all words over A (the free monoid on A)
A^+	the set of all non-empty words over A (the free semigroup on A)
$\langle\, A \mid R \,\rangle$	semigroup presentation with generating set A and defining relations R
C_n	the cyclic group of order n
$\mathcal{D}$	Green's relation $\mathcal{L} \vee \mathcal{R}$
$\mathcal{D}_a$	$\mathcal{D}$-class of a
D_n	dihedral group of order $2n$
$E(S)$	the set of idempotents of S
$\Gamma_r(H)$	Schützenberger group of the $\mathcal{H}$-class H

$G(P)$	the group defined by the presentation P
$\mathcal{H}$	Green's relation $\mathcal{L} \cap \mathcal{R}$
H_a	$\mathcal{H}$-class of a
$\mathcal{J}$	Green's relation defined by equality of principal ideals
J_a	$\mathcal{J}$-class of a
$\ker\ \alpha$	the kernel of the mapping α
$\mathcal{L}$	Green's relation defined by equality of principal left ideals
$\mathcal{L}_a$	$\mathcal{L}$-class of a
l.c.m.	the least common multiple
$L_1, L_2, \ldots$	layers number $1, 2, \ldots$
λ_u	left inner translation mapping of $\mathcal{S}$ by u
$\Im$	Rees congruence
N	normal form
$\mathbf{N}$	the set of natural numbers
Φ	the extension of $\phi : A \longrightarrow \mathcal{S}$ to the free semigroup A^+
$\mathcal{P}_n$	the presentation: $\langle a_1, \ldots, a_n : a_i^{m_i+1} = a_i, a_k a_j = a_j a_k a_j^{m_j}, 1 \leq i \leq n, 1 \leq k < j \leq n \rangle$
$\mathcal{R}$	Green's relation defined by equality of principal right ideals
R_a	$\mathcal{R}$-class of a
$\mathcal{R} \circ \mathcal{L}$	the composition of $\mathcal{R}$ and $\mathcal{L}$
$\mathcal{R} \vee \mathcal{L}$	the smallest equivalence relation containing $\mathcal{R}$ and $\mathcal{L}$
$\delta(\rho)$	the smallest congruence containing the binary relation ρ
ρ_u	right inner translation mapping of $\mathcal{S}$ by u

$\mathcal{S}$	semigroup
$SG(P)$	the semigroup defined by the presentation P
$\mathcal{S}^1$	semigroup with identity 1
$\lvert\mathcal{S}\rvert$	the order of the semigroup $\mathcal{S}$
$\mathcal{S}/\Im$	Rees quotient of $\mathcal{S}$ by $\Im$
$\mathcal{S}/\xi$	$\mathcal{S}$ factored by the equivalence relation ξ (the set of equivalence classes)
$\mathcal{S}_{(\ell,m,n)}$	the semigroup defined by the presentation: $\langle a,b,c : a^{\ell+1}=a, b^{m+1}=b, c^{n+1}=c, a^{\ell}ba=ab^{m-1}, a^{\ell}ca=ac^{n-1}, bc=cbc^{n}\rangle$
$\mathcal{S}_m$	the semigroup defined by the presentation: $\langle a,b,c : a^{3}=a, b^{m+1}=b, c^{m+1}=c, a^{2}ba=ab^{m-1}, a^{2}ca=ac^{m-1}, bc=cbc^{m}\rangle$
$\overline{\mathcal{S}}_n$	the semigroup defined by the presentation: $\langle a_1,\ldots,a_n : a_i^{m_i+1}=a_i, a_ka_j=a_ja_ka_j^{m_j}, 1\le i\le n, 1\le k<j\le n\rangle$
$T_\ell(H)$	the left stabilizer of the $\mathcal{H}$-class H
$T_r(H)$	the right stabilizer of the $\mathcal{H}$-class H
$[x]_\xi$	the equivalence class of x associated with the equivalence relation ξ
$V(a)$	the set of all inverses of a
$\xi^\natural$	natural homomorphism associated with the congruence ξ
$\mathbf{Z}$	the set of integers
$\mathbf{Z}_{p^t}$	the semigroup of integers modulo p^t where p is prime, with respect to multiplication

Chapter 1

Introduction

In this thesis we consider the following two fundamental problems of semigroup presentations :

1. Find a presentation defining a given semigroup $\mathcal{S}$.
2. Describe the semigroup defined by a given presentation $\langle A \mid \Re \rangle$.

We also establish other related results.

Early achievements in attacking the first problem include the work of Aĭzenštat [1, 2], Popova [39, 40], and Rédei [43]. The semigroups treated by Aĭzenštat [1] were the semigroup of all mappings of a finite set into itself under composition, the semigroup of all mappings of a countable set into itself which move only a finite number of elements, and the semigroup obtained from the latter by adjoining a mapping which permutes all the elements of the set cyclically. In each case, a presentation defining the semigroup considered was obtained. A presentation for the semigroup of order preserving transformations of a finite chain was obtained by Aĭzenštat in [2]. Popova [39, 40] obtained presentations for the semigroups of partial transformations of a finite set and partial endomorphisms of a finite chain. Rédei [43] proved that every finitely generated commutative semigroup is finitely presented.

More recently, presentations of alternating semigroups and the semigroup of all 2×2 matrices over the field of p elements have been given by Lipscomb [31] and Ruškuc [46] respectively. Campbell, Robertson, Ruškuc and Thomas [5, 6] described presentations of the Schützenberger group of the minimal two-sided ideal of semigroups having minimal two-sided ideals.

Achievements in attacking the second problem include the work of Robertson and Ünlü [44], Campbell, Robertson and Thomas [10, 11], and Campbell, Robertson, Ruškuc and Thomas [7, 9]. In [44], Robertson and Ünlü examined certain semigroup presentations, motivated by the fact that they, when considered as group presentations, define interesting groups. As a natural sequel of [44], Campbell, Robertson and Thomas [10] determined the cardinality and structure of the semigroups defined by the presentations:

$$\langle\, x_1, x_2, \ldots, x_n \mid x_i^{m+1} = x_i, x_j x_k^2 = x_k x_j^2 (1 \le i \le m, 1 \le j < k \le n \,\rangle.$$

In [9], the relationship between the groups and the semigroups defined by the Fibonacci group presentations:

$$\begin{gathered}\langle\, x_1, x_2, \ldots, x_n \mid x_1 x_2 \cdots x_r = x_{r+1}, x_2 x_3 \cdots x_{r+1} = x_{r+2}, \cdots, \\ x_{n-1} x_n x_1 \cdots x_{r-2} = x_{r-1}, x_n x_1 \cdots x x_{r-1} = x_r \,\rangle\end{gathered}$$

was considered along with the more general situation with the presentations:

$$\begin{gathered}\langle\, x_1, x_2, \ldots, x_n \mid x_1 x_2 \cdots x_r = x_{r+k}, x_2 x_3 \cdots x_{r+1} = x_{r+k+1}, \cdots, \\ x_{n-1} x_n x_1 \cdots x_{r+k-2} = x_{r+k-2}, x_n x_1 \cdots x x_{r-1} = x_{r+k-1} \,\rangle.\end{gathered}$$

In [7], presentations of Coxeter type for semigroups were defined and investigated. The authors showed that the minimal right ideals of a semigroup defined by such presentations are isomorphic to the group defined by the same presentation. They also described a necessary and sufficient condition for these semigroups to be finite. Then, the structure of semigroups defined by Coxeter type presentations for the symmetric and alternating groups was investigated in detail.

As an attempt to contribute to the achievements on the first problem, we obtain presentations for the commutative semigroups of integers, $\mathbf{Z}_{p^t}$, modulo p^t where p is prime and $t \in \mathbf{N}$ under multiplication.

Motivated by the work in [3, 4], we study the Schützenberger group of the $\mathcal{D}$-classes and obtain the results of Chapter 6 which can help in finding presentations of the Schützenberger groups for larger classes of semigroups.

In connection with the semigroup presentations examined by Robertson and Ünlü in [44], we consider the second problem and investigate generalized forms of one of these presentations. One of the presentations that they have examined was :

$$P \;=\; \langle\; a, b \mid a^3 = a, b^{m+1} = b, bab = a \;\rangle$$

which, when considered as a group presentation, defines the *dihedral* group D_n of order $2n$. They showed that P has a group kernel, that is, the semigroup defined by P has an ideal which is isomorphic to the group defined by P (which turned out to be the Schützenberger group of the minimal two-sided ideal). They used this fact to show that :

$$SG(P) = \{b, b^2, \ldots, b^n\} \cup K$$

where K is the group kernel. This method works for semigroups that have group kernels with complements that are describable. However, neither every group kernel has a describable complement, nor every semigroup has a group kernel. In fact the semigroup defined by :

$$\overline{P} \;=\; \langle\; a, b \mid a^3 = a, b^3 = b, ab^2 = b^2a \;\rangle$$

has no group kernel [44]. Motivated by these facts, we used different technique, described in 3.1, to examine two generalized forms of P :

$$P_1 = \;\langle\; a, b \mid a^3 = a, b^{m+1} = b, c^{m+1} = c, a^2ba = ab^{m-1}, a^2ca = ac^{m-1}, bc = cbc^m \;\rangle,$$

$P_2 = \langle\, a, b \mid a^{\ell+1} = a, b^{m+1} = b, c^{n+1} = c, a^{\ell}ba = ab^{m-1}, a^{\ell}ca = ac^{n-1}, bc = cbc^{n} \,\rangle.$

We study these semigroup presentations extensively in Chapter 4 and give a diagram relating their $\mathcal{D}$-classes in Chapter 5. Similar semigroup presentations, but more general, to the semigroup presentations in [10] are examined in Chapter 3.

The behaviour of the semigroups studied in Chapters 3 and 4 suggested the new semigroup classes, L-semi-commutative, R-semi-commutative and semi-commutative semigroups, which we define and study in Chapter 7. It turns out that these classes of semigroups are well-suited to the construction of fractions and archimedean semigroups. In fact their behaviour in this regard is almost the same as the behaviour of a commutative semigroup, as we will see in Chapter 7.

The author considers this work as an attempt to contribute to the development of semigroup presentations.

In the rest of this chapter, we state standard definitions and results that we use in the later chapters. Most of these results are well known and can be found in some introductory texts on semigroups, such as Lallement [30], Howie [23] and Grillet [19].

In this thesis, for non-zero integers x, and m, $x \pmod m$ is always in the range $1, \ldots, m$ unless stated otherwise.

1.1 Semigroups

Let $\mathcal{S}$ be a non-empty set, and let μ be a binary operation on $\mathcal{S}$, that is, a mapping $\mu : \mathcal{S} \times \mathcal{S} \longrightarrow \mathcal{S}$. We say that $(\mathcal{S}, \mu)$ is a *semigroup* if the binary operation μ is associative. Instead of writing $(\mathcal{S}, \mu)$ we write simply $\mathcal{S}$.

A semigroup $\mathcal{S}$ is called *commutative* if, for all $x, y \in \mathcal{S}$,

$$xy = yx.$$

If a semigroup $\mathcal{S}$ possesses an element 1 such that, for all $x \in \mathcal{S}$,

$$x1 = 1x = x,$$

then we say that 1 is an *identity element* of $\mathcal{S}$, and that $\mathcal{S}$ is a *semigroup with identity* or a *monoid.*

If $\mathcal{S}$ has no identity element, then one can easily adjoin an extra element 1 to $\mathcal{S}$ to form a monoid. We define 1 so that

$$(\forall x \in \mathcal{S})\ 1x = x1 = x \quad \text{and} \quad 1.1 = 1.$$

We then define

$$\mathcal{S}^1 = \begin{cases} \mathcal{S} & \text{if } \mathcal{S} \text{ has an identity element,} \\ \mathcal{S} \cup \{1\} & \text{otherwise.} \end{cases}$$

1.2 Congruences, homomorphisms, and ideals

Let $\mathcal{S}$ be a semigroup. A relation $\mathbf{R}$ on $\mathcal{S}$ is called *left compatible* if

$$(\forall x, y, a \in \mathcal{S})\ (x, y) \in \mathbf{R} \implies (ax, ay) \in \mathbf{R},$$

and *right compatible* if

$$(\forall x, y, a \in \mathcal{S})\ (x, y) \in \mathbf{R} \implies (xa, ya) \in \mathbf{R}.$$

It is called *compatible* if

$$(\forall x, y, s, t \in \mathcal{S})\ \left(\ (x, y) \in \mathbf{R} \text{ and } (s, t) \in \mathbf{R}\ \right) \implies (xs, yt) \in \mathbf{R}.$$

A left compatible equivalence relation is called a *left congruence.* A right compatible equivalence relation is called a *right congruence.* A compatible equivalence relation is called a *congruence.*

Let ρ be a congruence on a semigroup $\mathcal{S}$. Denote the set of equivalence classes by $\mathcal{S}/\rho$. Clearly $\mathcal{S}/\rho$ is a semigroup relative to the operation

$$[x][y] = [xy]$$

which is called the *quotient semigroup* of $\mathcal{S}$ by ρ.

Let $\mathcal{S}$ and T be two semigroups. A mapping $\psi : \mathcal{S} \longrightarrow T$ is a *homomorphism* if

$$(xy)\psi = (x)\psi(y)\psi$$

for all $x, y \in \mathcal{S}$. A homomorphism is called a *monomorphism* if it is injective. A surjective homomorphism is called an *epimorphism.* A bijective homomorphism is called an *isomorphism.* If $\mathcal{S} = T$ then any homomorphism is called an *endomorphism,* and any isomorphism is called an *automorphism.*

If ξ is a congruence on a semigroup $\mathcal{S}$, then $\mathcal{S}/\xi$ is a homomorphic image of $\mathcal{S}$ under the natural homomorphism $\xi^\natural$ which takes $s \in \mathcal{S}$ to $[s]_\xi$. Let $\phi : \mathcal{S} \longrightarrow T$ be a homomorphism. Then the *kernel*

$$\ker \phi = \{\ (x, y)\ :\ x, y \in \mathcal{S}, (x)\phi = (y)\phi\ \}$$

of ϕ is a congruence on $\mathcal{S}$ and one can easily show that $\mathcal{S}/\ker \phi \simeq \mathrm{Im}\ \phi$ and thus $\ker \xi^\natural = \xi$ and $(\ker \phi)^\natural = \phi$.

A subset I of a semigroup $\mathcal{S}$ is a *right ideal (left ideal)* if $is \in I$ $(si \in I)$ for all $i \in I$ and all $s \in \mathcal{S}$; equivalently, $I\mathcal{S} \subseteq I$ $(\mathcal{S}I \subseteq I)$. I is an *ideal* if it is both a right and a left ideal.

Proposition 1.2.1 [19, Proposition 1.4.6]. *Let I be an ideal of a semigroup $\mathcal{S}$. Then, the relation*

$$a \Im b \Longleftrightarrow a = b \text{ or } a, b \in I$$

is a congruence on $\mathcal{S}$.

The congruence $\Im$ is the *Rees congruence of the ideal I;* the quotient semigroup $\mathcal{S}/I = \mathcal{S}/\Im$ is the *Rees quotient of $\mathcal{S}$ by I.* Clearly, if $I = \emptyset$, then $\mathcal{S}/I = \mathcal{S}$.

1.3 Free semigroups

Let $\mathcal{S}$ be a semigroup, let A be a subset of $\mathcal{S}$, and let $\langle A\rangle$ be the intersection of all subsemigroups of $\mathcal{S}$ containing A. Clearly, $\langle A\rangle$ is a subsemigroup of $\mathcal{S}$ and $\langle A\rangle$ consists of all elements of $\mathcal{S}$ that can be expressed as finite products of elements in A.

Definition 1.3.1 Let $\mathcal{S}$ and A be as above. If $\langle A\rangle = \mathcal{S}$ we say that A is a set of *generators,* or a *generating set,* of $\mathcal{S}$.

Let $\mathbf{A}$ be a non-empty set, and let $\mathbf{A}^+$ stand for the set of all finite, non-empty words over $\mathbf{A}$. Define a binary operation on $\mathbf{A}^+$ by juxtaposition, that is,

$$(a_1a_2\ldots a_m)(b_1b_2\ldots b_n) = a_1a_2\ldots a_mb_1b_2\ldots b_n.$$

With respect to this operation, $\mathbf{A}^+$ is a semigroup, called the *free semigroup* on $\mathbf{A}$. According to Definition 1.3.1, the set $\mathbf{A}$ is a generating set for $\mathbf{A}^+$.

If we adjoin the identity 1 to $\mathbf{A}^+$ we obtain the *free monoid* on $\mathbf{A}$ which we denote by $\mathbf{A}^*$, and consider 1 as the *empty word.*

Proposition 1.3.1 [19, Proposition 1.5.4]. *Let $\mathcal{S}$ be a semigroup, and let A be a set. Then, every mapping f of A into $\mathcal{S}$ extends uniquely to a homomorphism ϕ of A^+ into $\mathcal{S}$. The image of ϕ is the subsemigroup of $\mathcal{S}$ generated by the image of f. If $\mathcal{S}$ is generated by the image of f, then ϕ is surjective.*

Proposition 1.3.1 implies that all semigroups generated by a set A are homomorphic images of A^+. Since every semigroup $\mathcal{S}$ is generated by some subset $A \subseteq \mathcal{S}$, we have the following proposition.

Proposition 1.3.2 *Every semigroup is a homomorphic image of a free semigroup.*

Definition 1.3.2 Let $\mathcal{S}$ be a semigroup, and let $x \in \mathcal{S}$. If $x^i = x^j$ for some distinct $i, j \in \mathbf{N}$, then the *index* of x is the least positive integer r such that $x^r = x^t$ holds in $\mathcal{S}$ for some $t > r$; the *period* of x is the least positive integer k such that $x^r = x^{r+k}$ holds in $\mathcal{S}$.

Proposition 1.3.3 (see [19]). *Let $\mathcal{S}$ be a semigroup, and let T be a cyclic semigroup generated by $x \in \mathcal{S}$. If for each $m, n \in \mathbf{N}$, $x^m = x^n$ implies that $m = n$, then T is isomorphic to the semigroup $(\mathbf{N}, +)$ of natural numbers with respect to addition. Otherwise, T is finite and there exist integers $r, k > 0$* (the *index* and *period* of x) *such that $x^{r+k} = x^r$ and every element of T can be written uniquely in the form x^i for some $i : 1 \leq i < r + k$. Furthermore, the set $\{ x^r, x^{r+1}, \ldots, x^{r+k-1} \}$ is a cyclic subgroup of T.*

Corollary 1.3.1 *Every non-empty finite semigroup contains an idempotent.*

1.4 Semigroup presentations

Let A be a non-empty set, let $\Re \subseteq A^+ \times A^+$, and let $\delta(\Re)$ be the smallest congruence on A^+ containing $\Re$. The ordered pair $\langle A \mid \Re \rangle$ is called a *semigroup presentation.* Each member of A is called *generating symbol,* and each member (a, b) of $\Re$, usually written $a = b$, is called a *defining relation.* A semigroup $\mathcal{S}$ is said to be *defined by* $\langle A \mid \Re \rangle$ if and only if $\mathcal{S} \cong A^+/\delta(\Re)$. A semigroup $\mathcal{S}$ is said to be *finitely presented* if it can be defined by a presentation $\langle A \mid \Re \rangle$ in which both A and $\Re$ are finite.

Let $\mathcal{S}$ be the semigroup defined by a presentation $\langle A \mid \Re \rangle$. If $u, v \in A^+$ and $(u, v) \in \delta(\Re)$, then we say that $u = v$ holds in $\mathcal{S}$ or $u = v$ is a *relation* in

$\mathcal{S}$ (we simply say $u = v$ in $\mathcal{S}$). If u and v are identical words, then we write $u \equiv v$.

A word $u \in A^+$ is said to be *directly derived* from $v \in A^+$ under $\Re$ if there exist $x, y \in A^*$ and $(a, b) \in \Re$ such that

$$(u \equiv xay,\ v \equiv xby) \quad \text{or} \quad (u \equiv xby,\ v \equiv xay).$$

A word $u \in A^+$ is said to be *derivable* from $v \in A^+$ under $\Re$ if for some $n \in \mathbf{N}$, there exist $z_1, \ldots, z_n \in A^+$ such that $u \equiv z_1,\ v \equiv z_n$ and z_{k+1} is directly derivable from z_k $(0 \leq k < n)$ under $\Re$. If u is derivable from v under $\Re$, then $(u, v) \in \delta(\Re)$ and thus we say $u = v$ is a *consequence* of $\Re$.

A slight modification of (Howie [23, Proposition 1.5.9]) gives the following proposition.

Proposition 1.4.1 *Let* $\mathbf{R}$ *be a relation on a semigroup* $\mathcal{S}$*, let* $\delta(\mathbf{R})$ *be the smallest congruence containing* $\mathbf{R}$*, and let* $a, b \in \mathcal{S}$*. Then* $(a, b) \in \delta(\mathbf{R})$ *if and only if either* $a = b$ *or,* b *is derivable from* a *under* $\mathbf{R}$*.*

Corollary 1.4.1 *Let* S *be the semigroup defined by a presentation* $\langle\ A \mid \Re\ \rangle$*, and let* $u, v \in A^+$*. Then,* $u = v$ *in* $\mathcal{S}$ *if and only if* v *is derivable from* u *under* $\Re$*.*

Example 1.4.1 The presentation $\langle\ a \mid a^{m+r} = a^r\ \rangle$ defines the finite cyclic semigroup $\mathcal{S}$ of index r and period m. ■

Example 1.4.2 The presentation $\langle\ a, b \mid a^2 = a, b^2 = b, ab = ba\ \rangle$ defines the commutative semigroup $\mathcal{S} = \{\ \{x\},\ \{y\},\ \{x, y\}\ \}$ $(x \neq y)$ under union. ■

Example 1.4.3 The presentation $\langle\ a, b \mid a^2 = a, b^2 = b\ \rangle$ defines an infinite semigroup as the subsemigroup generated by ab is infinite.

Proof. Assume that the semigroup defined by this presentation is finite. Then, there exist $m, n \in \mathbf{N}$ such that $(ab)^{m+n} = (ab)^n$. Since there is no relation in this presentation containing any power of ab and none of the generators is an identity, it follows that $(ab)^{m+n} = (ab)^n$ is not a consequence of the relations in this presentation which contradicts Corollary 1.4.1. Hence, the semigroup defined by this presentation is infinite. ∎

The definition of semigroup presentation can be easily applied to monoids with A^* replacing A^+.

Example 1.4.4 The presentation $\langle\ a, b \mid a = 1, b^2 = a\ \rangle$ defines the cyclic group C_2 of order 2. ∎

Example 1.4.5 The presentation $\langle\ a, b \mid a = 1, b^2 = b\ \rangle$ defines $\mathbf{Z}_2$, the semigroup (monoid) of integers modulo 2 under multiplication. ∎

The following proposition will be used in Chapter 2 .

Proposition 1.4.2 ([47]). *Let S be a semigroup generated by a set A, and let $\mathfrak{R} \subseteq A^+ \times A^+$. Then $\langle\ A \mid \mathfrak{R}\ \rangle$ is a presentation for S if and only if the following two conditions are satisfied:*

1. *S satisfies all the relations from $\mathfrak{R}$; and*
2. *if $u, v \in A^+$ are any words such that S satisfies the relation $u = v$, then $u = v$ is a consequence of $\mathfrak{R}$.*

1.5 Green's relations

In this section, we list the definitions of Green's relations along with some standard related results. Green's relations were first considered by J.A. Green [18] in 1951. They have played an important role in the development of semigroup theory.

Definition 1.5.1 An equivalence $\mathcal{R}$ on a semigroup $\mathcal{S}$ is defined by

$$a \ \mathcal{R} \ b \iff a\mathcal{S}^1 = b\mathcal{S}^1.$$

The set $a\mathcal{S}^1$ is called the *principal right ideal generated by* a.

Similarly, we define the equivalence $\mathcal{L}$ by

$$a \ \mathcal{L} \ b \iff \mathcal{S}^1 a = \mathcal{S}^1 b.$$

The set $\mathcal{S}^1 a$ is called the *principal left ideal generated by* a.

Proposition 1.5.1 *Let a, b be elements of a semigroup $\mathcal{S}$. Then $a \ \mathcal{R} \ b$ if and only if there exist $x, y \in \mathcal{S}^1$ such that $ax = b$, $by = a$. Also, $a \ \mathcal{L} \ b$ if and only if there exist $u, v \in \mathcal{S}^1$ such that $ua = b$, $vb = a$.*

Since the intersection of two equivalence relations is again an equivalence relation, the intersection of $\mathcal{R}$ and $\mathcal{L}$ is an equivalence relation denoted by $\mathcal{H}$. The equivalence relation $\mathcal{H}$ is of great importance in the development of the theory. The next proposition shows that $\mathcal{R} \circ \mathcal{L}$ is also an equivalence relation.

Proposition 1.5.2 (see [18]). *The equivalence relations $\mathcal{R}$ and $\mathcal{L}$ commute, that is,*

$$(\forall a, b \in \mathcal{S}) \ a \ \mathcal{R} \ x \ \mathcal{L} \ b \ \textit{for some} \ \ x \in \mathcal{S} \iff a \ \mathcal{L} \ y \ \mathcal{R} \ b \ \textit{for some} \ y \in \mathcal{S}.$$

Hence, the binary relation $\mathcal{D}$ defined by

$$\begin{aligned} a \ \mathcal{D} \ b &\iff a \ \mathcal{R} \ x \ \mathcal{L} \ b \quad \textit{for some} \quad x \in \mathcal{S} \\ &\iff a \ \mathcal{L} \ y \ \mathcal{R} \ b \quad \textit{for some} \quad y \in \mathcal{S} \end{aligned}$$

is an equivalence relation.

It then follows from (Howie [23 , Corollary 1.5.12]) that $\mathcal{R} \vee \mathcal{L} = \mathcal{R} \circ \mathcal{L}$.

The two-sided analogue of $\mathcal{R}$ and $\mathcal{L}$ is defined as follows:

Definition 1.5.2 An equivalence $\mathcal{J}$ on a semigroup $\mathcal{S}$ is defined by the rule

$$a \ \mathcal{J} \ b \Longleftrightarrow \mathcal{S}^1 a \mathcal{S}^1 = \mathcal{S}^1 b \mathcal{S}^1 \Longleftrightarrow (\exists x, y, u, v \in \mathcal{S}^1) \ : \ xay = b, \ ubv = a.$$

Clearly, $\mathcal{R} \subseteq \mathcal{J}$ and $\mathcal{L} \subseteq \mathcal{J}$. Hence, since $\mathcal{D}$ is the smallest equivalence relation containing $\mathcal{R}$ and $\mathcal{L}$, we must have $\mathcal{D} \subseteq \mathcal{J}$.

In groups and commutative semigroups we have

$$\mathcal{R} = \mathcal{L} = \mathcal{H} = \mathcal{D} = \mathcal{J};$$

and in a periodic semigroup we have $\mathcal{D} = \mathcal{J}$.

Notation 1.5.1 Let $\mathcal{S}$ be a semigroup and $a \in \mathcal{S}$. Then,

1. the $\mathcal{R}$-class of a is denoted by R_a,
2. the $\mathcal{L}$-class of a is denoted by $\mathcal{L}_a$,
3. the $\mathcal{D}$-class of a is denoted by $\mathcal{D}_a$,
4. the $\mathcal{H}$-class of a is denoted by H_a,
5. the $\mathcal{J}$-class of a is denoted by J_a.

It follows from the definition of $\mathcal{D}$ that $\mathcal{L}_a, \ R_a \subseteq \mathcal{D}_a$; and

$$a \ \mathcal{D} \ b \Longleftrightarrow R_a \cap \mathcal{L}_b \neq \emptyset \Longleftrightarrow R_b \cap \mathcal{L}_a \neq \emptyset.$$

Lemma 1.5.1 (Green's Lemma) *Let $\mathcal{S}$ be a semigroup. Assume that $a, b \in \mathcal{S}, \ u, v \in \mathcal{S}^1$, and $ua = b, \ vb = a$ so that $a \ \mathcal{L} \ b$. The left translations $\lambda_u : R_a \longrightarrow R_b$ which takes $x \in R_a$ to $ux \in R_b$ and $\lambda_v : R_b \longrightarrow R_a$, which takes $y \in R_b$ to $vy \in R_a$ are mutually inverse $\mathcal{L}$-class preserving bijections.*

Green's Lemma has a dual when $a\ \mathcal{R}\ b$ with the right translations, say, $\rho_{u'} : x \mapsto xu'$ and $\rho_{v'} : y \mapsto yv'$.

The next lemma follows from Green's Lemma and its dual.

Proposition 1.5.3 *Let a, b be elements in a semigroup $\mathcal{S}$. Then,*

1. *$a\ \mathcal{L}\ b$ implies that $|R_a| = |R_b|$ and $|H_a| = |H_b|$,*
2. *$a\ \mathcal{R}\ b$ implies that $|\mathcal{L}_a| = |\mathcal{L}_b|$ and $|H_a| = |H_b|$,*
3. *$a\ \mathcal{D}\ b$ implies that $|H_a| = |H_b|$.*

By Green's Lemma if $(as, a) \in \mathcal{R}$, then $x \mapsto xs$ is a bijection from H_a onto H_{as}. Hence, if $(as, a) \in \mathcal{H}$ then $x \mapsto xs$ is a bijection of H_a onto itself. This, plus the dual argument, gives the next lemma.

Lemma 1.5.2 *Let x, y be elements of a semigroup $\mathcal{S}$.*

1. *If $xy \in H_x$, then $\rho_y|_{H_x}$ is a bijection of H_x onto itself.*
2. *If $xy \in H_y$ then $\lambda_x|_{H_y}$ is a bijection of H_y onto itself.*

Hence, we have the following result.

Theorem 1.5.1 (Green's Theorem) *Let H be an $\mathcal{H}$-class in a semigroup $\mathcal{S}$. Then, either $H^2 \cap H = \emptyset$ or $H^2 = H$ and H is a subgroup of $\mathcal{S}$.*

The next result is an immediate consequence of Green's Theorem.

Corollary 1.5.1 *Let e be an idempotent in a semigroup $\mathcal{S}$. Then, H_e is a subgroup of $\mathcal{S}$. An $\mathcal{H}$-class in $\mathcal{S}$ contains at most one idempotent.*

Since the $\mathcal{H}$-classes are disjoint, the maximal subgroups of a semigroup $\mathcal{S}$ coincide with the $\mathcal{H}$-classes of $\mathcal{S}$ which contain idempotents. Each subgroup of $\mathcal{S}$ is contained in exactly one maximal subgroup of $\mathcal{S}$.

1.6 Regular $\mathcal{D}$-classes

An *inverse* of an element a of a semigroup $\mathcal{S}$ is an element b of $\mathcal{S}$ such that

$$aba = a \quad \text{and} \quad bab = b.$$

The set of all inverses of a is denoted by $V(a)$.

An element a of a semigroup $\mathcal{S}$ is *regular* if there exists $x \in \mathcal{S}$ such that $axa = a$.

Lemma 1.6.1 (see [19]). *For an element a of a semigroup $\mathcal{S}$ the following are equivalent:*

1. *a has an inverse;*
2. *a is regular;*
3. *R_a contains an idempotent;*
4. *$\mathcal{L}_a$ contains an idempotent.*

Corollary 1.6.1 *Let a be a regular element of a semigroup $\mathcal{S}$. Then every element of $\mathcal{D}_a$ is regular and every $\mathcal{R}$-class (and every $\mathcal{L}$-class) contained in $\mathcal{D}_a$ contains an idempotent.*

Proposition 1.6.1 (see [23]). *Every idempotent e in a semigroup $\mathcal{S}$ is a left identity for R_e and a right identity for $\mathcal{L}_e$.*

Proposition 1.6.2 (see [23]). *Let a be an element of a regular $\mathcal{D}$-class D in a semigroup $\mathcal{S}$, and let $a' \in V(a)$. Then,*

1. *$a' \in D$ and the two $\mathcal{H}$-classes $R_a \cap \mathcal{L}_{a'}$, $\mathcal{L}_a \cap R_{a'}$ contain the idempotents aa' and $a'a$, respectively,*

2. if $x \in \mathcal{H}_{a'} \cap V(a)$, then $x = a'$.

Proposition 1.6.3 (see [23]). *Let e, f be idempotents in a semigroup $\mathcal{S}$. Then $(e, f) \in \mathcal{D}$ if and only if there exist an element a in $\mathcal{S}$ and an inverse a' of a such that $aa' = e$, $a'a = f$.*

Proposition 1.6.4 (see [23]). *Let H and K be two regular $\mathcal{H}$-classes in the same $\mathcal{D}$-class. Then H and K are isomorphic.*

1.7 The semigroup enumeration program Semi

The Todd-Coxeter method for the enumeration of the cosets of a subgroup of a finitely presented group was one of the first applications of computers to abstract algebra in 1936 [51]. Recently, the related enumeration process for finitely presented semigroups has received some attention. In 1967 B.H. Neumann introduced an enumeration method for finitely presented semigroups related to the Todd-Coxeter coset enumeration process for groups [35]. A proof of the Neumann enumeration method was given by Jura in 1978 [26]. In 1992 E.F. Robertson and Y. Ünlü [44] described a machine implementation of a semigroup enumeration algorithm based on that of Neumann. T.G. Walker [52] working at St Andrews University described an implementation of a semigroup enumeration program which was called Semi. Since Semi was of very much help in our studies of semigroup presentations, it is worth reviewing some of the computations that Semi does.

Semi is a computer implementation of the semigroup analogue of the Todd-Coxeter process for groups. Given a presentation for a finite semigroup, it returns the order of the semigroup. Semi can also evaluate products, calculate ideals, take the union, intersection and complements of sets and determine whether a subset of a semigroup is a group. It can also evaluate the idempotents, provide a multiplication table and transformations representing elements of the semigroup and solve equations of the form

$$ax = b \text{ or } xa = b.$$

To enumerate a semigroup given by a presentation, enter the relations so that each relation is preceded by a slash (/) and separated by an equal (=) sign. When a number occurs it indicates that the immediately preceding item should be raised to that power. Thus the presentation

$$\langle\ a, b \mid a^3 = a, b^4 = b, (aba^2)^7 a = b\ \rangle$$

would be entered as

$$/a3 = a/b4 = b/(aba2)7a = b$$

followed by return.

A combination of Semi and GAP by S.A. Linton at the University of St Andrews has improved the performance of Semi and eased some of the complicated computations related to semigroup presentations such as finding the $\mathcal{D}$-classes, $\mathcal{H}$-classes and the other classes associated with Green's relations.

Chapter 2

The semigroup $\mathbf{Z}_{p^t}$

In the first section of this chapter we study Green's relations on the semigroup $\mathbf{Z}_{p^t}$ of integers modulo p^t, where p is prime and $t \in \mathbf{N}$ under multiplication. In the second section, we establish a presentation for the semigroup $\mathbf{Z}_{p^t}$ when p is an odd prime. In this chapter p stands for a prime number.

2.1 Green's relations on $\mathbf{Z}_{p^t}$

Since $\mathbf{Z}_{p^t}$ is finite and commutative, the $\mathcal{D}$-classes coincide with the $\mathcal{H}$-classes which coincide with the $\mathcal{R}$-classes and the $\mathcal{L}$-classes. Therefore, it is enough to study one of Green's relations and we choose the $\mathcal{R}$-relation. We first need the following technical lemmas.

Lemma 2.1.1 *If* $a \in \mathbf{Z}$ *and* $(a,p) = 1$, *then*

$$ia \equiv ja \pmod{p^t} \Longleftrightarrow i = j \ : \ 1 \le i,j \le p^t.$$

Corollary 2.1.1 *If* $x \in \mathbf{Z}_{p^t}$ *and* $(x,p) = 1$, *then* $\mathbf{Z}_{p^t}.x = \mathbf{Z}_{p^t}$.

Lemma 2.1.2 *If* $x = p^r y \in \mathbf{Z}_{p^t}$ *and* $(y,p) = 1$, *then, in* $\mathbf{Z}_{p^t}$ *we have*

$$\mathcal{R}_x = \{ z \in \mathbf{Z}_{p^t} : (z, p^t) = p^r \}.$$

Proof. It follows from Corollary 2.1.1 that

$$x\mathbf{Z}_{p^t} = p^r y \mathbf{Z}_{p^t} = p^r \mathbf{Z}_{p^t}.$$

Hence,

$$\mathcal{R}_x = p^r \mathbf{Z}_{p^t} \setminus p^{r+1} \mathbf{Z}_{p^t}. \qquad \blacksquare$$

Theorem 2.1.1 *The* $\mathcal{R}$*-classes of* $\mathbf{Z}_{p^t}$ *are :*

$$\mathcal{R}_1,\ \mathcal{R}_p,\ \mathcal{R}_{p^2},\ \mathcal{R}_{p^3}, \ldots,\ \mathcal{R}_{p^t}.$$

Proof. Corollary 2.1.1 implies that

$$\mathcal{R}_1 = \{ x \in \mathbf{Z}_{p^t} : (x, p) = 1 \}$$

and Lemma 2.1.2 implies that

$$\mathcal{R}_{p^i} = \{ x \in \mathbf{Z}_{p^t} : (x, p^t) = p^i \}. \qquad \blacksquare$$

Remark 2.1.1 Clearly $|\mathcal{R}_1| = p^t - p^{t-1}$ (see [45]), and

$|\mathcal{R}_{p^i}| = p^{t-i} - p^{t-i-1}$ $: 1 \le i \le t$ while $|\mathcal{R}_{p^t}| = 1$. Thus

$\sum_{i=0}^{t} |\mathcal{R}_{p^i}| = p^t - p^{t-1} + (\sum_{i=1}^{t-1} |\mathcal{R}_i|) + 1$

$= p^t - p^{t-1} + (p^{t-1} - p^{t-2} + p^{t-2} - p^{t-3} + \cdots + p - 1) + 1 = p^t.$

2.2 A presentation for $\mathbf{Z}_{p^t}$

In order to find a presentation for the semigroup $\mathbf{Z}_{p^t}$ we need to recall some definitions and results from number theory which we will mention here without proofs.

Definition 2.2.1 Let n be a positive integer. The *Euler ϕ-function* $\phi(n)$ is the number of positive integers not exceeding n which are relatively prime to n.

Definition 2.2.2 A *reduced residue system modulo* n is a set of $\phi(n)$ integers such that each element of the set is relatively prime to n, and no two different elements of the set are congruent modulo n.

Theorem 2.2.1 (Euler's Theorem). *Let m be a positive integer, and let a be an integer such that $(a, m) = 1$. Then,*

$$a^{\phi(m)} \equiv 1 \pmod{m}.$$

Definition 2.2.3 Let a and m be relatively prime positive integers. Then, the least positive integer x such that $a^x \equiv 1 \pmod{m}$ is called *the order of a modulo m.*

Notation 2.2.1 The order of a modulo m is denoted by $ord_m a$.

The following theorem helps us to find the solutions of the congruence

$$a^x \equiv 1 \pmod{m}.$$

Theorem 2.2.2 *Let a and m be relatively prime integers with $m > 0$, and let x be a positive integer. Then,*

$$a^x \equiv 1 \pmod{m} \Longleftrightarrow ord_m a \mid x.$$

Definition 2.2.4 Let r and m be relatively prime integers with $m > 0$. Then, r is called a *primitive root modulo* m if $ord_m r = \phi(m)$.

The following theorem shows one way in which primitive roots are useful.

Theorem 2.2.3 *Let* r *and* m *be relatively prime positive integers with* $m > 0$*, and let* r *be a primitive root modulo* m*. Then, the integers*

$$r^1,\ r^2,\ r^3, \ldots, r^{\phi(m)}$$

form a reduced residue set modulo m*.*

The next theorem shows that every power of an odd prime integer has at least one primitive root.

Theorem 2.2.4 (see [44]). *Let* p *be an odd prime. Then* p^k *has a primitive root for all positive integers* k*. Moreover, if* r *is a primitive root modulo* p^2*, then* r *is a primitive root modulo* p^k *for all positive integers* k*.*

Also, we need the following standard proposition.

Proposition 2.2.1 (see [47]). *Let* $\mathcal{S}$ *be a finite semigroup, let* A *be a generating set for* $\mathcal{S}$*, let* $\mathcal{K} \subseteq A^+ \times A^+$ *be a set of relations, and let* $W \subseteq A^+$*. Assume that the following conditions are satisfied:*

(1) the generators A *of* $\mathcal{S}$ *satisfy all the relations from* $\mathcal{K}$*;*
(2) for each word $w \in A^+$ *there exists a word* $\overline{w} \in W$ *such that* $w = \overline{w}$ *is a consequence of* $\mathcal{K}$*;*
(3) $|W| \leq |\mathcal{S}|$.

Then $\langle\ A \mid \mathcal{K}\ \rangle$ *is a presentation for* $\mathcal{S}$*.*

Proof. Let $w_1, w_2 \in A^+$ be any two words such that the relation $w_1 = w_2$ holds in $\mathcal{S}$. Condition (2) implies that there exist $\overline{w_1}, \overline{w_2} \in W$ such that the relations $w_1 = \overline{w_1}$ and $w_2 = \overline{w_2}$ are consequences of $\mathcal{K}$. By Condition (2) each element of $\mathcal{S}$ is represented by a word from W as a consequence of $\mathcal{K}$. Hence $|W| \geq |\mathcal{S}|$, so that (3) implies that $|W| = |\mathcal{S}|$, which means , since $\mathcal{S}$ is finite, that distinct elements of W represent distinct elements of $\mathcal{S}$.

Hence $w_1 = w_2$ is a consequence of $\mathcal{K}$. Since A generates $\mathcal{S}$ and A satisfies all the relations in $\mathcal{K}$, it follows from Proposition 1.4.2 that the presentation $\langle\, A \mid \mathcal{K} \,\rangle$ is a presentation for $\mathcal{S}$. ∎

The following lemmas gives us some insight on how to define a presentation for $\mathbf{Z}_{p^t}$.

Lemma 2.2.1 *Let* $y \in \mathbf{Z}_{p^t}$*, and let* x *be a primitive root modulo* p^t*. Then,*

1. $y.p^t \equiv p^t \pmod{p^t}$,
2. $x^{\phi(p^{t-i})}.p^i \equiv p^i \pmod{p^t}$ *for any* $i:\ 1 \leq i \leq\ t-1$.

Proof. 1. The proof is trivial,

2. by Euler's Theorem

$$x^{\phi(p^{t-i})} \equiv 1 \pmod{p^{t-i}}$$

which implies that

$$\exists\ q \in \mathbf{Z} : x^{\phi(p^{t-i})} = 1 + q.p^{t-i}.$$

Hence,

$$x^{\phi(p^{t-i})}.p^i = p^i + q.p^t. \tag{2.1}$$

It then follows from (2.4) that

$$x^{\phi(p^{t-i})}.p^i \equiv p^i \pmod{p^t}. \qquad \blacksquare$$

The next proposition gives a generating set for $\mathbf{Z}_{p^t}$.

Proposition 2.2.2 *The semigroup* $\mathbf{Z}_{p^t}$ *is generated by* p *and any primitive root* r *modulo* p^t*.*

Proof. Let $S(p,r)$ be the subsemigroup of $\mathbf{Z}_{p^t}$ generated by p and r. To show that $\mathbf{Z}_{p^t} = S(p,r)$, take $z \in \mathbf{Z}_{p^t}$. Then $z = fp^i$ for some integers $f,\ i\ : 1 \leq f < p^t,\ (f,p) = 1$ and $0 \leq i \leq t$. It then follows by Theorem 2.2.3 that

$$\exists\ k \in \mathbf{N} : (1 \leq k \leq \phi(p^t),\ f = r^k).$$

Thus,

$$z = r^k p^i : 1 \leq k \leq \phi(p^t),\ 0 \leq i \leq t$$

which implies that $z \in S(p,r)$ and $\mathbf{Z}_{p^t} \subseteq S(p,r)$. This shows that $\mathbf{Z}_{p^t} = S(p,r)$.

Now we are able to prove the following theorem which gives a presentation for the semigroup $\mathbf{Z}_{p^t}$ when p is an odd prime. In what follows p stands for an odd prime number.

Theorem 2.2.5 *Let p be an odd prime number greater than 1, and let $t \in \mathbf{N}$. Then, the semigroup $\mathbf{Z}_{p^t}$ has the following presentation:*

$$P = \langle\ a,\ b\ :\ a^{\phi(p^t)+1} = a, a^{\phi(p^t)}b = b,\ b^{t+1} = b^t, ab^t = b^t,\ ab = ba,$$
$$a^{\phi(p^{t-i})}b^i = b^i\ \ : 1 \leq i \leq t-1\ \rangle.$$

Proof. Proposition 2.2.2 says that $A = \{p, r\}$ is a generating set for $\mathbf{Z}_{p^t}$. Let

$$\mathcal{K} = \{(r^{\phi(p^t)+1}, r), (p^{t+1}, p^t), (pr, rp), (r^{\phi(p^t)}p, p),$$
$$(rp^t, p^t), (r^{\phi(p^{t-i})}p^i, p^i) : 1 \leq i \leq t-1\}.$$

Clearly $\mathcal{K} \subseteq A^+ \times A^+$. Lemma 2.2.1 and Theorem 2.2.1 imply that A satisfies all the relations in $\mathcal{K}$. Thus Condition (1) of Proposition 2.2.1 holds. Let W be the subset of A^+ defined by :

$$W = \{r^i, p^t : 1 \leq i \leq \phi(p^t)\} \cup (\cup_{j_0=1}^{t-1}\{r^i p^{j_0} : 1 \leq i \leq \phi(p^{t-j_0})\}. \qquad (2.2)$$

Since $rp = pr$ and $p^i = r^{\phi(p^{t-i})}p^i : 1 \leq i \leq t-1$, the relations in $\mathcal{K}$ imply that for each $w \in A^+$ there exists $\overline{w} \in W$ such that $w = \overline{w}$ is a consequence of $\mathcal{K}$. Hence Condition (2) of Proposition 2.2.1 holds. Finally, it follows from (2.5) that

$$|W| = \phi(p^t) + 1 + \sum_{j=1}^{t-1} \phi(p^{t-j})$$

$$= \phi(p^t) + 1 + p^{t-1} - 1 = p^t = |\mathbf{Z}_{p^t}|. \tag{2.3}$$

Now it follows from Proposition 2.2.1 that $\langle A \mid \mathcal{K} \rangle$ is a presentation for $\mathbf{Z}_{p^t}$ and thus P is a presentation for $\mathbf{Z}_{p^t}$. ■

Remark 2.2.1 The relation $a^{\phi(p^t)} = 1$ implies that $a^{\phi(p^t)}b = b$. On the other hand, since $\phi(p^{t-1})$ divides $\phi(p^t)$, the relation $a^{\phi(p^t)}b = b$ is a consequence of $a^{\phi(p^{t-i})}b^i = b^i$ when $i = 1$. Hence, the relation $a^{\phi(p^t)} = 1$ can be replaced by $a^{\phi(p^t)+1} = a$ and P becomes

$$P = \langle\, a,\, b \,:\, a^{\phi(p^t)+1} = a,\; b^{t+1} = b^t, ab^t = b^t,\; ab = ba,\; a^{\phi(p^{t-i})}b^i = b^i \;: 1 \leq i \leq t-1 \,\rangle.$$

What follows is an example of Semi, showing that the relations

$$a^{\phi(p^t)+1} = a,\; b^{t+1} = b^t,\; ab^t = b^t \;\text{ and }\; a^{\phi(p^{t-i})}b^i = b^i : 1 \;\leq i \leq t-1$$

are all necessary and thus, all the relations in P are necessary.

Example 2.2.1 Take $p = 3$ and $t = 4$. It follows that $\phi(p^t) = 54$, $|\mathbf{Z}_{p^t}| = 81$ and P in this case becomes

$$P = \langle\, a,\; b \;:\; a^{55} = a,\; b^4 = 0,\; ab = ba,\; a^{18}b = b,\; a^6b^2 = b^2,\; a^2b^3 = b^3 \,\rangle$$

which is equivalent to

$$P' = \langle\, a, b :\; a^{55} = a, b^5 = b^4, ab^4 = b^4, ab = ba, a^{18}b = b, a^6b^2 = b^2, a^2b^3 = b^3 \,\rangle$$

Since all the relations in P' with the exception of the first one have b on both sides, deleting the first relation gives an infinite semigroup with no finite period of a. Thus the first relation is necessary. To show that the second relation in P is necessary, we replace the relations $b^5 = b^4$ and $ab^4 = b^4$ of P' by $b^6 = b^5$ and $ab^5 = b^5$ respectively. The resulting semigroup in this case is of order 83. Hence the second relation in P is necessary. Also, deleting the fourth relation in P gives a semigroup of order 118, while deleting the fifth, or the sixth gives a semigroup of order 93 or 85 respectively. Hence, all the relations given in P are necessary.

The details of the computations applied to the above cases are on the next three pages.

Semigroup enumerator - T.G.W
Lookahead version 2.2 UNIX

e - Enumerate presentation

ef - Enumerate presentation from file

i - Find idempotents

l - List elements

w - Evaluate word

dw - Evaluate word, attempting to show derivation

id - Evaluate a left or right ideal sf - Print only element numbers when evaluating ideals

lf - Print element numbers and representations as words when evaluating ideals

t - Print definitions table

tf - Calculate transformations

ts - Alter table size

se - Solve equation

la - Set lookahead positions

cm - Select coincidence processing method

st - Store current subset

rs - Restore a stored subset

in - Intersect subsets

un - Take union of two subsets

gp - Determine whether current subset is a group

ps - Print current subset

cp - Complement current subset

h - Help

q - Quit

Table size set to 5000000.

semi > *e*

/a55=a/b5=b4/ab4=b4/ab=ba/a18b=b/a6b2=b2/a2b3=b3

```
Entering lookahead phase.
Lookahead done.
Packing table...
Packing done : 174 definitions recovered
Packing time : 0 user, 0 system, 0 total.

Order of semigroup is 81.
Defined : 256
Total enumeration time : 0 user, 0 system, 0 total.
semi > e
/a55=a/b6=b5/ab5=b5/ab=ba/a18b=b/a6b2=b2/a2b3=b3
Entering lookahead phase.
Lookahead done.
Packing table...
Packing done : 230 definitions recovered
Packing time : 0 user, 0 system, 0 total.

Order of semigroup is 83.
Defined : 314
Total enumeration time : 0 user, 0 system, 0 total.
semi > e
/a55=a/b5=b4/ab4=b4/ab=ba/a6b2=b2/a2b3=b3
Entering lookahead phase.
Lookahead done.
Packing table...
Packing done : 174 definitions recovered
Packing time : 0 user, 0 system, 0 total.

Order of semigroup is 118.
Defined : 293
Total enumeration time : 0 user, 0 system, 0 total.
```

```
semi> e
/a55=a/b5=b4/ab4=b4/ab=ba/a18b=b/a2b3=b3
Entering lookahead phase.
Lookahead done.
Packing table...
Packing done : 168 definitions recovered
Packing time : 0 user, 0 system, 0 total.

Order of semigroup is 93.
Defined : 262
Total enumeration time : 0 user, 0 system, 0 total.
semi> e
/a55=a/b5=b4/ab4=b4/ab=ba/a18b=b/a6b2=b2
Entering lookahead phase.
Lookahead done.
Packing table...
Packing done : 167 definitions recovered
Packing time : 0 user, 0 system, 0 total.

Order of semigroup is 85.
Defined : 253
Total enumeration time : 0 user, 0 system, 0 total.
semi> q
```

Chapter 3

Semigroups related to the direct product of cyclic groups

In this chapter we consider semigroup presentations of the form

$$\mathcal{P}_n = \langle a_1, a_2, \ldots, a_n : a_i^{m_i+1} = a_i, a_k a_j = a_j a_k a_j^{m_j}, 1 \leq i \leq n,\ 1 \leq k < j \leq n \rangle$$

where $m_i, n \in \mathbf{N},\ 1 \leq i \leq n$.

As before, $G(\mathcal{P}_n)$ and $SG(\mathcal{P}_n)$ stand for the group and semigroup defined by $\mathcal{P}_n$ respectively. Clearly,

$$G(\mathcal{P}_n) = C_{m_1} \times C_{m_2} \times \cdots \times C_{m_n}$$

where C_{m_i} is the cyclic group of order m_i.

In the first section of this chapter we introduce the technique we use to study semigroups defined by presentations. In the second and third sections we use that technique to show that the semigroups $SG(\mathcal{P}_n)$, corresponding to the finite direct products of finite cyclic groups $G(\mathcal{P}_n)$ are also finite and give general formulas for their orders after we find normal forms for them. In the fourth section we consider the idempotents of such semigroups and give the sets of idempotents of $SG(\mathcal{P}_2)$ and $SG(\mathcal{P}_3)$. In the fifth section we concentrate on $SG(\mathcal{P}_2)$ and $SG(\mathcal{P}_3)$ and find the maximal subgroups, which they contain, along with their presentations.

So, let us first introduce the technique that allows us to do all of that.

3.1 The technique

The technique we use in this chapter and in the next chapter is quite simple but very powerful in investigating semigroups defined by presentations. It depends on the following fact about semigroups:

> Let $\langle A \mid \Re \rangle$ and $\langle B \mid \Re' \rangle$ be semigroup presentations satisfying that there exists a bijection mapping $f : A \longrightarrow B$ such that replacing the symbols in $\Re$ by their image under f gives a set of relations contained in $\Re'$. Then the semigroup defined by $\langle B \mid \Re' \rangle$ is a homomorphic image of the semigroup defined by $\langle A \mid \Re \rangle$ (Proposition 3.2.1).

We summarize this technique in the following steps :

1. deduce fundamental relations from the given presentation $\langle A \mid \Re \rangle$,
2. use the relations found in 1 to find the smallest possible subset N of A^+ that satisfies the first condition of Definition 3.2.1 below ,
3. choose a set of transformations B defined on a set of integers X with a set of relations $\Re' \subseteq B^+ \times B^+$ such that there is a bijection $f : A \longrightarrow B$ satisfying the condition that the image of $\Re$ under f is contained in $\Re'$ and the semigroup defined by $\langle B \mid \Re' \rangle$ gives a tool to show that the words of N are distinct elements in the semigroup $SG(\langle\ A \mid \Re \rangle)$.

We demonstrate this technique in this chapter and will realize its wide use in the next chapter.

3.2 Normal form for $SG(\mathcal{P}_2)$

In this section we will find a normal form for the semigroup defined by

$$\mathcal{P}_2 = \langle a, b : a^{m_1+1} = a, b^{m_2+1} = b, ab = bab^{m_2} \rangle.$$

and find a formula for its order. But let us first agree on what we mean by a normal form.

Definition 3.2.1 Let S be a semigroup defined by a presentation $\langle A \mid \mathfrak{R} \rangle$. A subset N of A^+ is called a *normal form* for S if and only if the following two conditions are satisfied:

1. each word $w \in A^+$ can be transformed to a word $w^* \in N$ by applying relations from $\mathfrak{R}$,
2. if $u, v \in N, u \not\equiv v$, then $u \neq v$ in S.

Example 3.2.1 Let S be the semigroup defined by the presentation

$$\langle a, b : a^2 = a, b^2 = b, ab = ba \rangle.$$

Then the set $\{a, b, ab\}$ is a normal form for S while the set $\{a, b, ab, ba\}$ is not, because $ab \not\equiv ba$ and $ab = ba$ in S.

The following lemma gives some useful properties of $SG(\mathcal{P}_2)$.

Lemma 3.2.1 *If a and b are as in $\mathcal{P}_2$, then*

$$ba^j b = a^j b^2 : 1 \leq j \leq m_1.$$

Proof. We use mathematical induction. When $j = 1$, we get

$$bab = bab^{m_2+1} = \underbrace{bab^{m_2}}b = \underbrace{ab}b = ab^2.$$

This shows that the lemma is true when $j = 1$.

We now assume that the lemma is valid for all positive integers j with $j \leq k$ for some $k \leq m_1 - 1$. Hence, we have

$$\begin{aligned} ba^{k+1}b &= ba^k(ab) = ba^k(bab^{m_2}) = \underbrace{ba^kb}\,ab^{m_2} \\ &= \underbrace{a^kb^2}\,ab^{m_2} \qquad \text{(by assumption)} \\ &= a^kb(bab^{m_2}) = a^kb(ab^{m_2+1}) \qquad \text{(by assumption)} \\ &= a^kb(ab) = a^k\,\underbrace{bab} = a^k\,\underbrace{ab^2} = a^{k+1}b^2. \end{aligned}$$

This establishes the lemma. ∎

Our aim is to prove the following theorem.

Theorem 3.2.1 *The semigroup $SG(\mathcal{P}_2)$ has the following normal form :*

$$N = \{a^i, b^j, a^ib^j, b^ja^i, a^ib^ja^k : 1 \leq i, k \leq m_1, 1 \leq j \leq m_2\}.$$

To prove this theorem we need to show that N satisfies the conditions of Definition 3.2.1. The first condition follows by Lemma 3.2.1. To show that N satisfies the second condition, we will define two mappings α and β satisfying the relations of $\mathcal{P}_2$ and allowing us to show that the words in N represent distinct elements in $SG(\mathcal{P}_2)$. So, we will come back to the proof of this theorem after we define the required mappings.

Choose two distinct prime numbers p_1 and p_2 greater than 3. Let

$$A = \{2^i.3^jp_1 : 0 \leq i \leq m_1, 0 \leq j \leq m_2\},$$

$$B = \{2^i.3^jp_2 : 0 \leq i \leq m_1, 0 \leq j \leq m_2\}.$$

Take $X = A \cup B$. Define $\alpha, \beta : X \longrightarrow X$ as follows:

$$(x)\alpha = 2x,$$

$$(x)\beta = \begin{cases} 3x : x \in A, \\ p_2 : x \in B. \end{cases}$$

where the non-zero powers of 2 and 3 are reduced modulo m_1 and m_2 respectively.

Remark 3.2.1 We agree that, unless stated otherwise, $x \pmod{m_i}$ is in the range $1, \ldots, m_i$ for all $x \in \mathbf{N}, 1 \leq i \leq 2$. Also, as mentioned above, the non-zero powers of 2 and 3 are reduced modulo m_1 and m_2 respectively.

In the next lemma we verify that α and β satisfy the relations of $\mathcal{P}_2$.

Lemma 3.2.2 *The mappings α and β satisfy the following:*

(1) $\alpha^{m_1+1} = \alpha$,

(2) $\beta^{m_2+1} = \beta$,

(3) $\alpha\beta = \beta\alpha\beta^{m_2}$.

Proof. (1) Let $x \in X$. Then

$$(x)\alpha^{m_1+1} = 2^{m_1+1}x = 2x = (x)\alpha. \tag{3.1}$$

(2) As in (1).

(3) Let $x \in X$. Then

$$(x)\beta\alpha\beta^{m_2} = \begin{cases} 3.2.3^{m_2}x \ (= 2.3x) : x \in A, \\ p_2 : x \in B, \end{cases} \tag{3.2}$$

and

$$(x)\alpha\beta = \begin{cases} 2.3x : x \in A, \\ p_2 : x \in B, \end{cases}. \tag{3.3}$$

i.e. $\alpha\beta = \beta\alpha\beta^{m_2}$. eprf

The next proposition enables us to show that the semigroup generated by α and β, under composition, is a homomorphic image of $SG(\mathcal{P}_2)$.

Proposition 3.2.1 (Van Dyck) *Let $\langle A \mid \Re \rangle$ and $\langle B \mid \Re' \rangle$ be semigroups presentations. Assume that the following condition is satisfied:*

There exists a bijection $f : A \longrightarrow B$ such that replacing all the symbols in $\Re$ by their images under f gives a set of relations contained in $\Re'$.

Then the semigroup defined by $\langle B \mid \Re' \rangle$ is a homomorphic image of the semigroup defined by $\langle A \mid \Re \rangle$.

Proof. Let S and S^* be the semigroups defined by $\langle A \mid \Re \rangle$ and $\langle B \mid \Re' \rangle$ respectively. Thus, there exist isomorphisms $\psi : A^+/\rho \longrightarrow S$ and $\psi^* : B^+/\rho^* \longrightarrow S^*$ where ρ and ρ^* are the smallest congruences containing $\Re$ and $\Re'$ respectively. Extend f to $\phi : A^+ \longrightarrow B^+$ as follows:

$$(x_1 x_2 \cdots x_r)\phi = (x_1)f(x_2)f \cdots (x_n)f : x_i \in A.$$

Clearly ϕ is an isomorphism. Let $\phi^* : A^+/\rho \longrightarrow B^+/\rho^*$ be defined by

$$([x]_\rho)\phi^* = [(x)\phi]_{\rho^*}$$

where $[x]_\rho$ and $[x]_{\rho^*}$ are the equivalent classes of x with respect to the congruences ρ and ρ^*. By Proposition 1.4.1 $(x, y) \in \rho$ implies that $((x)\phi, (y)\phi) \in \rho^*$. Hence, ϕ^* is a well-defined homomorphism. Also ϕ^* is onto because if $[w] \in B/\rho^*$ then $[(w)\phi^{-1}] \in A/\rho$ and $([(w)\phi^{-1}])\phi^* = [w]$. Now take $\theta = \psi^{-1} o \phi^* o \psi^* : S \longrightarrow A^+/\rho \longrightarrow B/\rho^* \longrightarrow S^*$. Since ψ, ϕ^* and ψ^* are onto homomorphism, θ is an onto homomorphism from S to S^*. ■

Remark Note that it is enough to have f surjective in Proposition 3.2.1.

Corollary 3.2.1 *The semigroup generated by α and β, with respect to composition, is a homomorphic image of $SG(\mathcal{P}_2)$.*

Let $S(\alpha, \beta)$ be the semigroup generated by α and β with respect to composition, and let N' be the image of the set N, given in Theorem 3.2.1, under the homomorphic extension ϕ of $f : \{a, b\} \longrightarrow \{\alpha, \beta\}$ which takes a to α and b to β. The following proposition shows that some of the elements in N' are distinct in $S(\alpha, \beta)$.

Proposition 3.2.2 *Let Ω_1 and Ω_2 be the subsets of N' defined by:*

$$\Omega_1 = \{\alpha^i, \beta^j, \alpha^i\beta^j, \beta^r\alpha^d : 1 \leq i, d \leq m_1, 1 \leq j, r \leq m_2\},$$

$$\Omega_2 = \{\alpha^i\beta^j, \alpha^k\beta^r\alpha^d : 1 \leq i, d, k \leq m_1, 1 \leq j, r \leq m_2\}.$$

Then for each $i \in \{1, 2\}$, the elements of Ω_i are distinct in $S(\alpha, \beta)$.

Proof. Clearly $\alpha^i \neq \beta^j$ for any i and j because $(p_1)\alpha^i = 2^i \neq 3^j = (p_1)\beta^j$. Also $\alpha^i\beta^j \neq \alpha^k$ for any $i, j, k \geq 1$ because

$$(p_1)\alpha^i\beta^j = 2^i.3^j p_1 \neq 2^k p_1 = (p_1)\alpha^k.$$

Similarly, $\alpha^i\beta^j \neq \beta^r$ and $\alpha^k \neq \beta^j\alpha^i \neq \beta^r$ for any $i, j, k, r \geq 1$. Since the relation $\alpha^i\beta^j = \alpha^k\beta^r$ implies that:

$$2^i.3^j p_1 = (p_1)\alpha^i\beta^j = (p_1)\alpha^k\beta^r = 2^k.3^r p_1,$$

it follows that

$$\alpha^i\beta^j = \alpha^k\beta^r \iff i = k, j = r.$$

Similarly,

$$\beta^j\alpha^i = \beta^r\alpha^k \iff j = r, i = k.$$

Now assume that $\alpha^i\beta^j = \beta^r\alpha^d$ for some $i, j, d, r \geq 1$. Then

$$p_2 = (p_2)\alpha^i\beta^j = (p_2)\beta^r\alpha^d = 2^d p_2$$

which is a contradiction. Thus $\alpha^i\beta^j \neq \beta^r\alpha^k$ for any i, j, k, r.

So far, we have shown that

$$\alpha^k \neq \alpha^i\beta^j \neq \beta^r, \tag{3.4}$$

$$\alpha^k \neq \beta^j\alpha^i \neq \beta^r, \tag{3.5}$$

$$\alpha^i\beta^j \neq \beta^r\alpha^k, \tag{3.6}$$

$$\alpha^i\beta^j = \alpha^k\beta^r \iff i = k, j = r \tag{3.7}$$

and

$$\beta^j\alpha^i = \beta^r\alpha^k \iff i = k, j = r. \tag{3.8}$$

Thus, (3.4),(3.5),(3.6),(3.7) and (3.8) establish that the elements of Ω_1 are distinct in $S(\alpha,\beta)$. To show that the elements of Ω_2 are distinct in $S(\alpha,\beta)$, it is enough to show that $\alpha^i\beta^j \neq \alpha^k\beta^r\alpha^d$ for any i, j, k, r, d and that the relation $\alpha^i\beta^j\alpha^k = \alpha^{i'}\beta^{j'}\alpha^{k'}$ holds if and only if $i = i', j = j'$ and $k = k'$. So assume that $\alpha^i\beta^j = \alpha^k\beta^r\alpha^d$. Then

$$p_2 = (p_2)\alpha^i\beta^j = (p_2)\alpha^k\beta^r\alpha^d = 2^d p_2$$

which is a contradiction. Hence,

$$\alpha^i\beta^j \neq \alpha^k\beta^r\alpha^d. \tag{3.9}$$

Also $\alpha^i\beta^j\alpha^k = \alpha^{i'}\beta^{j'}\alpha^{k'}$ implies that

$$2^{i+k}.3^j p_1 = 2^{i'+k'}.3^j p_1 \Longrightarrow i + k \equiv i' + k' \pmod{m_1},\ j = j'$$

and

$$2^k p_2 = 2^{k'} p_2 \Longrightarrow k = k' \Longrightarrow i = i'.$$

Thus

$$\alpha^i\beta^j\alpha^k = \alpha^{i'}\beta^{j'}\alpha^{k'} \iff i = i', j = j', k = k'. \tag{3.10}$$

Thus, (3.7),(3.9) and (3.10) establish that the elements of Ω_2 are distinct in $S(\alpha,\beta)$.

Remark 3.2.2 The following table summarizes the actions of N' on p_1 and p_2 and gives the other conditions:

No	w	action on p_1	action on p_2	related to
1	α^i	$2^i p_1$	$2^i p_2$	
2	β^j	$3^j p_1$	p_2	
3	$\alpha^i \beta^j$	$2^i.3^j p_1$	p_2	
4	$\beta^j \alpha^i$	$2^i.3^j p_1$	$2^i p_2$	
5	$\alpha^i \beta^j \alpha^k$	$2^{i+k}.3^j p_1$	$2^k p_2$	4

Table 3.1:The actions of N' on p_1 and p_2.

Corollary 3.2.2 *If a and b are as in $\mathcal{P}_2$, and $1 \leq i, k, d \leq m_1$, $1 \leq j, r \leq m_2$, then:*

(1) The elements $a^i, b^j, a^i b^j, b^r a^k$ are distinct in $SG(\mathcal{P}_2)$;

(2) the elements $a^i b^j, a^k b^r a^d$ are distinct in $SG(\mathcal{P}_2)$.

Proof. The result follows by Corollary 3.2.1 and Proposition 3.2.2. ∎

Lemma 3.2.3 *If a and b are as in $\mathcal{P}_2$, then*

$$a^i b^j a^k \neq b^r a^d \text{ for any } 1 \leq i, k, d \leq m_1, 1 \leq j, r \leq m_2.$$

Proof. Assume that $a^i b^j a^k = b^r a^d$ for some $i, j, k, r, d \geq 1$. Then multiplying both sides by a from the left implies that $a^{i+1} b^j a^k = a b^r a^d$, and (3.10) implies that $i = m_1$, $j = r$ and $k = d$. Thus

$$a^{m_1} b^j a^k = b^j a^k. \tag{3.11}$$

Multiplying both sides of (3.11) by b^{m-j+1} from the left and by a^{m-k+1} from the right gives

$$a^{m_1} ba = ba \text{ (Lemma 3.2.1).} \tag{3.12}$$

But since none of the relations in $\mathcal{P}_2$ has ba as one of its sides (3.12) is not a consequence of any of the relations in $\mathcal{P}_2$ and therefore $a^i b^j a^k \neq b^r a^d$. ■

Now we go back to prove Theorem 3.2.1.

Proof of Theorem 3.2.1 The first condition of Definition 3.2.1 follows by Lemma 3.2.1 . Also, Corollary 3.2.2, Lemma 3.2.3 and the fact that

$$(p_1)\alpha^i = 2^i p_1 \neq 2^i.3^j.2^k p_1 = (p_1)\alpha^i\beta^j\alpha^k \neq 3^j p_1 = (p_1)\beta^j$$

imply that the words in N represent distinct elements in $SG(\mathcal{P}_2)$. ■

Theorem 3.2.2 *The order of $SG(\mathcal{P}_2)$ is given by the following formula:*

$$|SG(\mathcal{P}_2)| = m_1 + m_2(m_1+1)^2. \tag{3.13}$$

Proof. The result follows by counting the words in N as they represent distinct elements in $SG(\mathcal{P}_2)$. ■

3.3 Normal form for $SG(\mathcal{P}_n) : n \geq 3$

In this section we generalize the results of Section 1 to the more general case $SG(\mathcal{P}_n)$ where

$$\mathcal{P}_n = \langle\, a_1, a_2, \ldots, a_n : a_i^{m_i+1} = a_i, a_j a_k = a_k a_j a_k^{m_k} : 1 \leq i \leq n, 1 \leq j < k \leq n \,\rangle.$$

Remark 3.3.1 In the rest of this chapter, the non-zero powers of $a_1, a_2, \ldots, a_n$ are reduced modulo $m_1,\ m_2, \ldots, m_n$ respectively and, unless stated otherwise, $x \pmod{m_i}$ is in the range $1, \ldots, m_i$ for all $x \in \mathbf{N}, 1 \leq i \leq n$.

We start with the following lemma which exhibits some useful properties of $SG(\mathcal{P}_3)$ where we choose a_1, a_2 and a_3 to be a, b and c respectively. Hence,

$$\mathcal{P}_3 = \langle\, a, b, c : a^{m_1+1} = a, b^{m_2+1} = b, c^{m_3+1} = c, ab = bab^{m_2}, ac = cac^{m_3}, bc = cbc^{m_3} \,\rangle.$$

Lemma 3.3.1 *Let a, b, and c be as in $\mathcal{P}_3$, and let $1 \le i, k \le m_2$, $1 \le d, r \le m_3$, $1 \le j, t \le m_1$. Then,*

1. $b^i a^j b^k = a^j b^{i+k}$,
2. $c^d a^j c^r = a^j c^{d+r}$,
3. $c^d b^i c^r = b^i c^{d+r}$,
4. $ca^j b^i c = a^j b^i c^2$,
5. $cb^i a^j c = b^i a^j c^2$,
6. $ca^j b^i a^t c = a^j b^i a^t c^2$.

Proof. The proofs for (1), (2) and (3) are as in Lemma 3.2.1;

(4) by (3), $b^i c = c^{m_3} b^i c$, and thus

$$\begin{aligned} ca^j(b^i c) &= ca^j(c^{m_3} b^i c) = (ca^j c^{m_3}) b^i c \\ &= (a^j c) b^i c \quad (\text{ by } (2)\ ) \\ &= a^j b^i c^2 \quad (\text{by } (3)\ ); \end{aligned}$$

(5) by (2), $a^j c = c^{m_3} a^j c$, and thus

$$\begin{aligned} cb^i a^j c &= cb^i(c^{m_3} a^j c) = b^i c a^j c \quad (\text{ by } (3)\ ) \\ &= b^i a^j c^2 \quad (\text{by } (2)\ ); \end{aligned}$$

(6) follows from (4) and (2). ■

Corollary 3.3.1 *If a, b and c are as in $\mathcal{P}_3$, then*

$$(1) \qquad c^d a^j b^i c^r = a^j b^i c^{d+r},$$

$$(2) \qquad c^d b^i a^j c^r = b^i a^j c^{d+r},$$

$$(3) \qquad c^d a^j b^i a^t c^r = a^j b^i a^t c^{d+r}.$$

Lemma 3.3.1 and Corollary 3.3.1 can be generalized to $SG(\mathcal{P}_n)$ as in the following two lemmas and their corollaries.

Lemma 3.3.2 *For a given integer n, the following relations hold in $SG(\mathcal{P}_n)$*

$$a_j a_i^t a_j = a_i^t a_j^2 : 1 \le i, j \le n, \ \ 1 \le t \le m_i, \ i < j.$$

Proof. When $t = 1$ and $i < j$ we get

$$\begin{aligned} a_j a_i a_j &= (a_j a_i a_j^{m_j}) a_j \\ &= a_i a_j a_j = a_i a_j^2. \end{aligned}$$

This shows that the lemma is true when $t = 1$.

We now assume that the lemma is true for all positive integers t with $t \le k$ for some integer $k \le m_i - 1$. Hence, we have

$$\begin{aligned} a_j a_i^{k+1} a_j &= a_j a_i^k (a_i a_j) = a_j a_i^k (a_j a_i a_j^{m_j}) \\ &= (a_j a_i^k a_j) a_i a_j^{m_j} = (a_i^k a_j^2) a_i a_j^{m_j} \qquad \text{(by assumption.)} \\ &= a_i^k (a_j^2 a_i a_j^{m_j}) = a_i^{k+1} a_j^2. \qquad \text{(by assumption.)} \qquad \blacksquare \end{aligned}$$

Corollary 3.3.2 *In $SG(\mathcal{P}_n)$ where n is a given integer, we have*

$$a_j^p a_i^t a_j^q = a_i^t a_j^{p+q} \quad \textit{whenever} \quad i < j.$$

We now use Lemma 3.3.2 and Corollary 3.3.2 to prove the following more general lemma.

Lemma 3.3.3 *If $a_{i_1}^{t_1} a_{i_2}^{t_2} \cdots a_{i_k}^{t_k}$ is any word of the symbols $a_1, a_2, \ldots, a_k$ where t_j, $(1 \le j \le k)$, might be zero and $n > k \ge 1$, then*

$$a_{k+1}a_{i_1}^{t_1}a_{i_2}^{t_2}\cdots a_{i_k}^{t_k}\ a_{k+1} = a_{i_1}^{t_1}a_{i_2}^{t_2}\cdots a_{i_k}^{t_k}a_{k+1}^2.$$

Proof. By Corollary 3.3.2,

$$\begin{aligned}
&a_{k+1}a_{i_1}^{t_1}a_{i_2}^{t_2}\cdots a_{i_k}^{t_k}\ a_{k+1} = a_{k+1}a_{i_1}^{t_1}a_{i_2}^{t_2}\cdots a_{i_{k-1}}^{t_{k-1}}\ a_{k+1}a_{i_k}^{t_k}a_{k+1}^{m_{k+1}}\\
&= a_{k+1}a_{i_1}^{t_1}a_{i_2}^{t_2}\cdots a_{i_{k-2}}^{t_{k-2}}\ a_{k+1}a_{i_{k-1}}^{t_{k-1}}\ a_{k+1}^{m_{k+1}}a_{i_k}^{t_k}a_{k+1}^{m_{k+1}} = \cdots\\
&= a_{k+1}a_{i_1}^{t_1}a_{k+1}a_{i_2}^{t_2}a_{k+1}^{m_{k+1}}\ a_{i_3}^{t_3}a_{k+1}^{m_{k+1}}\cdots a_{i_k}^{t_k}a_{k+1}^{m_{k+1}}\\
&= a_{i_1}^{t_1}a_{k+1}^2a_{i_2}^{t_2}a_{k+1}^{m_{k+1}}a_{i_3}^{t_3}a_{k+1}^{m_{k+1}}\cdots a_{i_k}^{t_k}a_{k+1}^{m_{k+1}}\\
&= a_{i_1}^{t_1}a_{i_2}^{t_2}a_{k+1}^2a_{i_3}^{t_3}a_{k+1}^{m_{k+1}}\cdots a_{i_k}^{t_k}a_{k+1}^{m_{k+1}} = \cdots\\
&= a_{i_1}^{t_1}a_{i_2}^{t_2}\cdots a_{i_k}^{t_k}a_{k+1}^2. \qquad \blacksquare
\end{aligned}$$

Corollary 3.3.3 *If* $w = a_{i_1}^{t_1}a_{i_2}^{t_2}\cdots a_{i_k}^{t_k}$, *then* $a_{k+1}^{i}wa_{k+1}^{j} = wa_{k+1}^{i+j}$.

Notation 3.3.1 In the rest of this chapter we let $\overline{\mathcal{S}}_k$ stand for $SG(\mathcal{P}_k)$ and define $N(\overline{\mathcal{S}}_k)$, inductively, by :

$$N(\overline{\mathcal{S}}_k) = N(\overline{\mathcal{S}}_{k-1})\bigcup_{i=1}^{m_k}\{\ a_k^i,\ a_k^i x,\ xa_k^i,\ xa_k^i y : x, y, z \in N(\overline{\mathcal{S}}_{k-1})\ \}.$$

with $N(\overline{\mathcal{S}}_2) = N$ (see Theorem 3.2.1).

The next theorem shows that $N(\overline{\mathcal{S}}_n)$ is a normal form for $\overline{\mathcal{S}}_n$ whenever $2 \leq n,\ n \in \mathbf{N}$.

Theorem 3.3.1 *Let $n \in \mathbf{N}$ such that $n \geq 2$. Then, the semigroup $\overline{\mathcal{S}}_n$ has $N(\overline{\mathcal{S}}_n)$ as a normal form.*

Proof. We use mathematical induction. It follows from Theorem 3.2.1 that it is true when $n = 2$. We now assume that the theorem is valid for all positive integers j with $2 \leq j \leq k$ for some integer k. Let $A_i = \{ a_1, a_2, \ldots, a_i \}$. Then, by Corollary 3.3.3 it follows that in $\overline{\mathcal{S}}_{k+1}$,

$$A_{k+1}^+ = (A_k \cup \{a_{k+1}\})^+ = A_k^+ \bigcup_{x \in \{a_{k+1}\}^+} \{ x,\ xA_k^+,\ A_k^+x,\ A_k^+xA_k^+ \}. \quad (3.14)$$

It then follows from Definition 3.2.1 and our assumption that for each $x \in A_k^+$ there exists $y \in N(\overline{\mathcal{S}}_k)$ such that $x = y$ in $\overline{\mathcal{S}}_k$ and thus in $\overline{\mathcal{S}}_{k+1}$ (as $\overline{\mathcal{S}}_k \subset \overline{\mathcal{S}}_{k+1}$). Since for each $x \in \{a_{k+1}\}^+$ there exists $y \in \{a_{k+1}^i : 1 \leq i \leq m_{k+1}\}$ such that $x = y$ in $\overline{\mathcal{S}}_{k+1}$, it follows from (3.14) and the fact that $\overline{\mathcal{S}}_k \subset \overline{\mathcal{S}}_{k+1}$ that

$$\forall x \in A_{k+1}^+, \exists y \in (N(\overline{\mathcal{S}}_k) \bigcup_{i=1}^{m_{k+1}} \{ a_{k+1}^i,\ a_{k+1}^i N(\overline{\mathcal{S}}_k),\ N(\overline{\mathcal{S}}_k) a_{k+1}^i N(\overline{\mathcal{S}}_k) \}) = N(\overline{\mathcal{S}}_{k+1})$$

such that $x = y$ in $\overline{\mathcal{S}}_{k+1}$. This establishes the first condition of Definition 3.2.1, that is,

$$\forall w \in A_{k+1}\ (\exists\ w' \in N(\overline{\mathcal{S}}_{k+1}) : w = w' \text{ in } \overline{\mathcal{S}}_{k+1}).$$

Next, we show that different elements in $N(\overline{\mathcal{S}}_{k+1})$ represent different elements in $\overline{\mathcal{S}}_{k+1}$. Clearly, $x \not\equiv y$ for any $x, y \in N(\overline{\mathcal{S}}_{k+1})$. Since $x \not\equiv y$ in $N(\overline{\mathcal{S}}_k)$ implies that $x \neq y$ in $\overline{\mathcal{S}}_k$, it follows that $x \neq y$ in $\overline{\mathcal{S}}_{k+1}$ as $\overline{\mathcal{S}}_k$ is contained in $\overline{\mathcal{S}}_{k+1}$. Since none of the words

$$a_{k+1}^{m_{k+1}+t}\ ,\ a_{k+1}^r w a_{k+1}^t : 1 \leq r, t \leq m_{k+1}, w \in N(\overline{\mathcal{S}}_k)$$

is in $N(\overline{\mathcal{S}}_{k+1})$, it follows that the relations in $\mathcal{P}_{k+1}$ have no effect on any two different words in $N(\overline{\mathcal{S}}_{k+1})$ and thus $x \not\equiv y$ in $N(\overline{\mathcal{S}}_{k+1})$ implies that $x \neq y$ in $\overline{\mathcal{S}}_{k+1}$. This shows that the second condition of Definition 3.2.1 holds. Hence, $N(\overline{\mathcal{S}}_{k+1})$ is a normal form for $\overline{\mathcal{S}}_{k+1}$ and the result is true for each integer $n \geq 1$. ∎

Alternatively, one can show that different elements in $N(\overline{\mathcal{S}}_{k+1})$ represent different elements in $\overline{\mathcal{S}}_{k+1}$, with the assumption that it is true for $N(\overline{\mathcal{S}}_k)$ and

$\overline{\mathcal{S}}_k$, as follows:

Let p be a prime number greater than 2, let

$$X = \{\ 2^i p : 0 \le i \le m_{k+1}\},$$

$$T = \{a_{k+1}^i, wa_{k+1}^i, a_{k+1}^i w, za_{k+1}^i w : 1 \le i \le m_{k+1}, z, w \in N(\overline{\mathcal{S}}_k)\},$$

and let $\alpha_i : X \longrightarrow X : 1 \le i \le k+1$ be defined as follows:

$$(x)\alpha_i = \begin{cases} x : 1 \le i \le k, \\ 2x : i = k+1. \end{cases}$$

Let $\overline{\mathcal{S}}_{\alpha_{(k+1)}}$ be the semigroup generated by these mappings with respect to composition. Clearly, $\overline{\mathcal{S}}_{\alpha_{(k+1)}}$ is a homomorphic image of $\overline{\mathcal{S}}_{k+1}$. These mappings and the fact that $\overline{\mathcal{S}}_{\alpha_{(k+1)}}$ is a homomorphic image of $\overline{\mathcal{S}}_{k+1}$ imply that if $x \in N(\overline{\mathcal{S}}_k)$ and $y \in T$, then $x \ne y$ in $\overline{\mathcal{S}}_{k+1}$. Hence, it remains to show that different elements in T represent different elements in $\overline{\mathcal{S}}_{k+1}$. So, let $\beta : A_{k+1} \longrightarrow \overline{\mathcal{S}}_k$ be defined by:

$$(a_i)\beta = \begin{cases} a_i : 1 \le i \le k, \\ a_1 : i = k+1. \end{cases}$$

Let ϕ be the homomorphic extension of β to $\mathcal{S}_{k+1}^+ \longrightarrow \overline{\mathcal{S}}_k$. Clearly ϕ is onto. Assume that $a_{k+1}^i w = za_{k+1}^j : z, w \in N(\overline{\mathcal{S}}_k)$ and that either w doesn't end with a_1 or z doesn't start with a_1. Then, $\overline{\mathcal{S}}_{\alpha_{(k+1)}}$ and ϕ imply that $a_1 w = za_1$ in $\overline{\mathcal{S}}_k$. Thus, there are three cases to be examined:

1. both $a_1 w$ and za_1 are in $N(\overline{\mathcal{S}}_k)$,
2. only one of them is in $N(\overline{\mathcal{S}}_k)$,
3. none of them is in $N(\overline{\mathcal{S}}_k)$.

(Note that if $w \in N(\overline{\mathcal{S}}_i) : 1 \le i \le k$, then there exists $w' \in N(\overline{\mathcal{S}}_i) \cap \{a_2, a_3, \ldots, a_i\}^+$ such that $a_1 w a_1 = a_1^t w' a_1^r \in N(\overline{\mathcal{S}}_i)$ for some $r, t : 1 \le r, t \le m_1$).

The first case implies that $a_1 w \equiv za_1$ which is impossible. For the second case, we may assume that $a_1 w \in N(\overline{\mathcal{S}}_k)$. Then,

$$za_1 = z_0a_1^t : z_0, z_0a_1^t \in N(\overline{\mathcal{S}}_k).$$

Thus, $a_1w \equiv z_0a_1^t$ which is impossible. For the third case we will have $a_1w = a_1^tw_0, za_1 = z_0a_1^s : w_0, z_0, a_1^tw_0, z_0a_1^s \in N(\overline{\mathcal{S}}_k)$ and thus $z_0a_1^s \equiv a_1^tw_0$ which is impossible as well. On the other hand, if w ends with a_1 and z starts with a_1, then we modify β as follows:

$$(a_i)\beta = \begin{cases} a_i : 1 \leq i \leq k, \\ a_2 : i = k+1. \end{cases}$$

and then each of the above cases will lead to a contradiction. Hence,

$$a_{k+1}^iw \neq za_{k+1}^j \quad \text{for any} \quad w, z \in N(\overline{\mathcal{S}}_k), 1 \leq i, j \leq m_{k+1}. \tag{3.15}$$

Assume that $xa_{k+1}^i = ya_{k+1}^jz$ for some $x, y, z \in N(\overline{\mathcal{S}}_k), 1 \leq i, j \leq m_{k+1}$. Then, $\overline{\mathcal{S}}_{\alpha_{(k+1)}}$ and ϕ imply that $xa_1 = ya_1z$ in $\overline{\mathcal{S}}_k$. As before, there are three cases to be examined and we can argue as above to show that each case leads to a contradiction (remember that if $ya_1z \notin N(\overline{\mathcal{S}}_k)$ then there exists $y_0, z_0 \in N(\overline{\mathcal{S}}_k)$ such that $ya_1z = y_0a_1^tz_0$, and $y_0a_1^tz_0 \in N(\overline{\mathcal{S}}_k)$). Thus,

$$xa_{k+1}^i \neq ya_{k+1}^jz \quad \text{for any} \quad x, y, z \in N(\overline{\mathcal{S}}_k), 1 \leq i, j \leq m_{k+1}. \tag{3.16}$$

Similarly,

$$a_{k+1}^ix \neq ya_{k+1}^jz \quad \text{for any} \quad x, y, z \in N(\overline{\mathcal{S}}_k), 1 \leq i, j \leq m_{k+1}. \tag{3.17}$$

Now we show that if $x, y, z, w \in N(S_k), 1 \leq i, j \leq m_{k+1}$, then

$$xa_{k+1}^iy = za_{k+1}^jw \Longleftrightarrow x \equiv z,\ y \equiv w,\ i = j.$$

So, assume that $xa_{k+1}^iy = za_{k+1}^jw$. Then, $\overline{\mathcal{S}}_{\alpha_{(k+1)}}$ implies that $i = j$ and thus

$$xa_{k+1}^iy = za_{k+1}^iw \tag{3.18}$$

multiplying both sides of (3.18) by a_{k+1}^{m-i+1} from the left gives $xa_{k+1}y = za_{k+1}w$. Hence, $xa_{k+1}y$ can be changed to $za_{k+1}w$ after a finite number of applications of the relations in $\mathcal{P}_{k+1}$ to $xa_{k+1}y$. But the relations in $\mathcal{P}_{k+1}$ act only when we have words contain parts of the form $a_k\overline{w}a_k$ where a_d is involved in $\overline{w}$ if and only if $d \le k$, which is not the case in $xa^i_{k+1}y$ and $za^i_{k+1}w$ as $x, y, z, w \in N(\overline{\mathcal{S}}_k)$. Hence, if $x, y, z, w \in N(S_k), 1 \le i, j \le m_{k+1}$, then

$$xa^i_{k+1}y = za^j_{k+1}w \Longleftrightarrow x \equiv z,\ y \equiv w,\ i = j. \tag{3.19}$$

Similarly, one can show that if $x, y \in N(\overline{\mathcal{S}}_k)$ and $1 \le i, j \le m_{k+1}$, then

$$xa^i_{k+1} = ya^j_{k+1} \Longleftrightarrow i = j, x \equiv y, \tag{3.20}$$

$$a^i_{k+1}x = a^j_{k+1}y \Longleftrightarrow i = j, x \equiv y. \tag{3.21}$$

This concludes that distinct elements in $N(\overline{\mathcal{S}}_{k+1})$ represent distinct elements in $\overline{\mathcal{S}}_{k+1}$ and thus $N(\overline{\mathcal{S}}_{k+1})$ is a normal form for $\overline{\mathcal{S}}_{k+1}$ and the result of this theorem holds. ■

Theorem 3.3.2 *For a given integer n, the order of $\overline{\mathcal{S}}_n$ is given by the following formula:*

$$|\overline{\mathcal{S}}_n| = |\overline{\mathcal{S}}_{n-1}| + m_n(|\overline{\mathcal{S}}_{n-1}| + 1)^2$$

Proof. The result of this theorem follows immediately by counting the elements of $N(\overline{\mathcal{S}}_n)$. To see this fact more closely, let $X_n = N(\overline{\mathcal{S}}_n) \setminus N(\overline{\mathcal{S}}_{n-1})$. It then follows from Theorem 3.3.1 that if $w \in X_n$ then $w = w_1a^i_nw_2$ where $w_1, w_2 \in N(\overline{\mathcal{S}}_{n-1}) \cup (\overline{\mathcal{S}}^1_n \setminus \overline{\mathcal{S}}_n), 1 \le i \le m_n$. Assume that the order of $\overline{\mathcal{S}}_{n-1}$ is L and that

$$N(\overline{\mathcal{S}}_{n-1}) = \{w_1, w_2, w_3, \ldots, w_L\}.$$

We display X_n in such a way that we can count its elements. So we display X_i as in the following table (recall that $\overline{\mathcal{S}}_k$ stands for $SG(\mathcal{P}_k)$):

a^i_n	$w_1a^i_n$	$w_2a^i_n$	$w_3a^i_n$	$\cdots$	$w_La^i_n$
$a^i_nw_1$	$w_1a^i_nw_1$	$w_2a^i_nw_1$	$w_3a^i_nw_1$	$\cdots$	$w_La^i_nw_1$
$a^i_nw_2$	$w_1a^i_nw_2$	$w_2a^i_nw_2$	$w_3a^i_nw_2$	$\cdots$	$w_La^i_nw_2$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$a^i_nw_L$	$w_1a^i_nw_L$	$w_2a^i_nw_L$	$w_3a^i_nw_L$	$\cdots$	$w_La^i_nw_L$

By counting the elements of X_n we find that

$$|X_n| = m_n(L+1)^2 = m_n(|SG(\mathcal{P}_2)|+1)^2.$$

It then follows that

$$|SG(\mathcal{P}_n)| = |SG(\mathcal{P}_{n-1})| + m_n(|SG(\mathcal{P}_{n-1})|+1)^2. \blacksquare$$

Example 3.3.1 Consider the semigroup $\overline{\mathcal{S}}_3$ with a_1, a_2, a_3 replaced by a, b, c respectively. Then $\overline{\mathcal{S}}_3$ has the following normal form:

$$N(\overline{\mathcal{S}}_3) = N(SG(\mathcal{P}_2)) \bigcup_{k=1}^{m_3} X_{(3,k)}$$

where $X_{(3,k)}$ is as displayed in the following table:

c^k	a^ic^k	b^jc^k	$a^ib^jc^k$	$b^ja^ic^k$	$a^ib^ja^dc^k$
c^ka^i	$a^ic^ka^d$	$b^jc^ka^i$	$a^ib^jc^ka^d$	$b^ja^ic^ka^d$	$a^ib^ja^dc^ka^t$
c^kb^j	$a^ic^kb^j$	$b^jc^kb^t$	$a^ib^j\ c^kb^t$	$b^ja^ic^kb^t$	$a^ib^ja^dc^kb^t$
$c^ka^ib^j$	$a^ic^ka^db^j$	$b^jc^ka^ib^t$	$a^ib^jc^ka^rb^t$	$b^ja^ic^ka^db^t$	$a^ib^ja^dc^ka^rb^t$
$c^kb^ja^i$	$a^dc^kb^ja^i$	$b^tc^kb^ja^i$	$a^db^tc^kb^ja^i$	$b^ta^dc^kb^ja^i$	$a^db^ta^rc^kb^ja^i$
$c^ka^ib^ja^d$	$a^rc^ka^ib^ja^d$	$b^tc^ka^ib^ja^d$	$a^rb^tc^ka^ib^ja^d$	$b^ta^rc^ka^ib^ja^d$	$a^rb^ta^lc^ka^ib^ja^d$

The words of $X_{(3,k)} : 1 \leq i,\ d,\ r,\ l\ \leq\ m_1,\ 1 \leq\ j,\ t\ \leq\ m_2.$

In the next section we consider the idempotents of $SG(\mathcal{P}_n) : n \geq 2$.

3.4 The sets of idempotents of $SG(\mathcal{P}_n) : n \geq 2$

In this section we will give a necessary and sufficient condition for $w \in \overline{\mathcal{S}}_n$ to be an idempotent. Then, we apply that condition to find the sets of idempotents of $\overline{\mathcal{S}}_2$ and $\overline{\mathcal{S}}_3$.

Lemma 3.4.1 *Let $x, y \in N(\overline{\mathcal{S}}_{k-1}) \cup (\overline{\mathcal{S}}_k^1 \setminus \overline{\mathcal{S}}_k)$, and let $i \in \mathbf{N} : 1 \leq i \leq m_k$. Then xa_k^iy is an idempotent in $\overline{\mathcal{S}}_k$ if and only if $i = m_k$ and $xyx = x$.*

Proof. Let $\overline{1}$ stands for the identity of $\overline{\mathcal{S}}^1_k$. If $x = y = \overline{1}$, then the result is immediate. So, assume that $x \neq \overline{1}$ or $y \neq \overline{1}$ and that xa^i_ky is an idempotent. Then,

$$xa^i_ky = (xa^i_ky)(xa^i_ky) = xyxa^{2i}_ky \quad \text{(by Lemma 3.3.3)}. \tag{3.22}$$

Since $xyx \in \overline{\mathcal{S}}_{k-1}$, there exists $z \in N(\overline{\mathcal{S}}_{k-1})$ such that $xyx = z$ in $\overline{\mathcal{S}}_{k-1}$. Hence,

$$xa^i_ky = za^{2i}_ky : x, y, z \in \ N(\overline{\mathcal{S}}_{k-1}) \cup (\overline{\mathcal{S}}^1_k \setminus \overline{\mathcal{S}}_k).$$

Also, there exists $r : 1 \leq r \leq m_k$ such that $a^{2i}_k = a^r_k$ in $\overline{\mathcal{S}}_k$. Thus,

$$xa^i_ky = za^r_ky : x, y, z \in \ N(\overline{\mathcal{S}}_{k-1}), 1 \leq i, r \leq m_k.$$

It then follows from (3.19) ((3.20) when $y = \overline{1}$) that $x \equiv z$ and $i = r$. Therefore,

$$x \equiv z = xyx, \ a^i_k = a^{2i}_k$$

concluding that $xyx = x$ and $2i \equiv i \pmod{m_k}$. Since $m_k \geq i$, we must have $i = m_k$. Conversely, if $i = m_k$ and $xyx = x$, then

$$(xa^{m_k}_ky)^2 = xyxa^{m_k}_ky = xa^{m_k}_ky. \qquad \blacksquare$$

Corollary 3.4.1 *Let $x, y \in N(\overline{\mathcal{S}}_{k-1}) \cup (\overline{\mathcal{S}}^1_k \setminus \overline{\mathcal{S}}_k)$, and assume that $xa^{m_k}_ky$ is an idempotent. Then xy and yx are idempotents.*

Proof. By Lemma 3.4.1 $xyx = x$. Hence,

$$(xy)^2 = xyxy = (xyx)y = xy, \quad (yx)^2 = y(xyx) = yx. \qquad \blacksquare$$

Remark 3.4.1 1. Note that Lemma 3.4.1 implies that if x is the identity of $\overline{\mathcal{S}}^1_k$, then y is the identity of $\overline{\mathcal{S}}^1_k$ as well (i.e. $x = y = \overline{1}$).

2. Recall that in $\mathcal{P}_2$ and $\mathcal{P}_3$ we chosed a_1, a_2 and a_3 to be a, b and c respectively, and that all the powers of a, b and c are reduced modulo m_1, m_2 and m_3 respectively. Also, unless stated otherwise, $x \pmod{m_i}$ is in the range $1, \ldots, m_i$ for all $x \in \mathbf{N}$.

3. We agree that if $x = m_i$ for some $i : 1 \leq i \leq n$, then $a^{m_i - x}_i = a^{m_i}_i$.

Lemma 3.4.2 *The set of idempotents of* $\overline{\mathcal{S}}_2$ *is :*

$$\{a^{m_1},\ b^{m_2},\ a^{m_1}b^{m_2},\ a^{m_1}b^{m_2}a^{m_1}\ a^ib^{m_2}a^{m_1-i}\ :\ 1 \leq i\ \leq m_1 - 1\ \}.$$

Proof. First, note that a^{m_1} is an idempotent in $\overline{\mathcal{S}}_2$. Let $w \in \overline{\mathcal{S}}_2 \setminus \overline{\mathcal{S}}_1$. Then, by Theorem 3.2.1 there exists $\overline{w} \in N(\overline{\mathcal{S}}_2)$ such that $w = \overline{w}$ in $\overline{\mathcal{S}}_2$ and

$$w = \overline{w} \equiv xb^dy : x = a^i, y = a^j, 1 \leq d \leq m_2, 0 \leq i, j \leq m_1.$$

By Lemma 3.4.1 w is an idempotent if and only if $xyx = x$ and $d = m_2$. Hence,

$$xyx = a^ia^ja^i = a^{2i+j} = x = a^i \iff 2i + j \equiv i \pmod{m_1}. \qquad (3.23)$$

It then follows from Theorem 3.2.1, Lemma 3.4.1, (3.23) and Remark 3.4.1 that the set of idempotents in $\overline{\mathcal{S}}_2$ is:

$$\{\ a^{m_1}, b^{m_2}, a^{m_1}b^{m_2}, a^{m_1}b^{m_2}a^{m_1}, a^ib^{m_2}a^{m_1-i} : 1 \leq i \leq m_1 - 1\ \} \qquad \blacksquare$$

Similarly, we find the set of idempotents of $\overline{\mathcal{S}}_3$ which we display, in alphabetic order, in the table given in the next lemma for the sake of comparison with what we have in the proof.

Lemma 3.4.3 *The set of idempotents of* $\overline{\mathcal{S}}_3$ *is as displayed in the following table :*

a^{m_1}	$a^ib^ja^dc^{m_3}a^kb^{m_2-j}a^{m_1-i-d-k}$	$a^ib^jc^{m_3}a^kb^{m_2-j}a^{m_1-i-k}$
$a^{m_1}b^{m_2}$	$a^ib^ja^{m_1-i}c^{m_3}b^{m_2-j}$	$a^{m_1}b^jc^{m_3}b^{m_2-j}$
$a^ib^{m_2}a^{m_1-i}$	$a^ib^ja^dc^{m_3}b^{m_2-j}a^{m_1-i-d}$	$a^ib^jc^{m_3}b^{m_2-j}a^{m_1-i}$
$a^ib^{m_2}a^{m_1-i}c^{m_3}$	$a^{m_1}b^{m_2}c^{m_3}$	$a^{m_1}c^{m_3}, a^ic^{m_3}a^{m_1-i}$
$a^ib^{m_2}a^kc^{m_3}a^{m_1-i-k}$	$a^ib^{m_2}c^{m_3}a^{m_1-i}$	$b^{m_2}, b^{m_2}c^{m_3}$
$a^ib^ja^dc^{m_3}a^{m_1-i-d}b^{m_2-j}$	$a^ib^jc^{m_3}a^{m_1-i}b^{m_2-j}$	$b^jc^{m_3}b^{m_2-j},\ c^{m_3}$

Table 3.5:The set of idempotents of $\overline{\mathcal{S}}_3 :\ 1 \leq i, k, d \leq m_1,\ 1 \leq j \leq m_2.$

Proof. First, note that the idempotents of $\overline{\mathcal{S}}_2$ are idempotents in $\overline{\mathcal{S}}_3$. Let $w \in \overline{\mathcal{S}}_3 \setminus \overline{\mathcal{S}}_2$. Then, by Theorem 3.3.1, there exists $\overline{w} \in N(\overline{\mathcal{S}}_3)$ such that $w = \overline{w}$ in $\overline{\mathcal{S}}_3$ and

$$w = \overline{w} \equiv xc^d y : x, y \in N(\overline{\mathcal{S}}_2) \cup (\overline{\mathcal{S}}_3^1 \setminus \overline{\mathcal{S}}_3), 1 \leq d \leq m_3.$$

By Lemma 3.4.1 w is an idempotent if and only if $d = m_3$ and $xyx = x$. Lemma 3.3.3, Example 3.3.1 and Remark 3.4.1 imply that x and y, in order to satisfy this condition, must be in one of the forms given in the next table with ($\surd$) in the corresponding rectangle.

y	x	a^i	b^j	a^ib^j	b^ja^i	$a^ib^ja^k$
$\overline{\mathcal{S}}_3^1 \setminus \overline{\mathcal{S}}_3$	$*$	$\surd$	$\surd$	$\surd$	$\times$	$\surd$
a^i	$*$	$\surd$	$\times$	$\surd$	$\times$	$\surd$
b^j	$*$	$\times$	$\surd$	$\surd$	$\times$	$\surd$
a^ib^j	$*$	$\times$	$\times$	$\surd$	$\times$	$\surd$
b^ja^i	$*$	$\times$	$\times$	$\surd$	$\times$	$\surd$
$a^ib^ja^k$	$*$	$\times$	$\times$	$\surd$	$\times$	$\surd$

Hence, the last table, Remark 3.4.1, Corollary 3.4.1 and Lemma 3.4.2 imply that if $x, y \in N(\overline{\mathcal{S}}_2)$ and $w = xc^{m_3}y$ is an idempotent, then the forms of $\overline{w}$, relative to the rows of the above table, are:

1. $a^{m_1}c^{m_3}$, $b^{m_2}c^{m_3}$, $a^{m_1}b^{m_2}c^{m_3}$, $a^{m_1}b^{m_2}a^{m_1}c^{m_3}$, $a^ib^{m_2}a^{m_1-i}c^{m_3}$;

2. $a^{m_1}c^{m_3}a^{m_1}$, $a^ic^{m_3}a^{m_1-i}$, $a^{m_1}b^{m_2}c^{m_3}a^{m_1}$, $a^ib^{m_2}c^{m_3}a^{m_1-i}$, $a^{m_1}b^{m_2}a^{m_1}c^{m_3}a^{m_1}$, $a^ib^{m_2}a^kc^{m_3}a^{m_1-i-k}$;

3. $b^{m_2}c^{m_3}b^{m_2}$, $b^jc^{m_3}b^{m_2-j}$, $a^{m_1}b^{m_2}c^{m_3}b^{m_2}$, $a^{m_1}b^jc^{m_3}b^{m_2-j}$, $a^{m_1}b^{m_2}a^{m_1}c^{m_3}b^{m_2}$, $a^ib^ja^{m_1-i}c^{m_3}b^{m_2-j}$;

4. $a^{m_1}b^{m_2}c^{m_3}a^{m_1}b^{m_2}$, $a^ib^jc^{m_3}a^{m_1-i}b^{m_2-j}$, $a^{m_1}b^{m_2}a^{m_1}c^{m_3}a^{m_1}b^{m_2}$, $a^ib^ja^kc^{m_3}a^{m_1-i-k}b^{m_2-j}$;

5. $a^{m_1}b^{m_2}c^{m_3}b^{m_2}a^{m_1}$, $a^ib^jc^{m_3}b^{m_2-j}a^{m_1-i}$, $a^{m_1}b^{m_2}a^{m_1}c^{m_3}b^{m_2}a^{m_1}$, $a^ib^ja^kc^{m_3}b^{m_2-j}a^{m_1-i-k}$;

6. $a^{m_1}b^{m_2}c^{m_3}a^{m_1}b^{m_2}a^{m_1},\ a^ib^jc^{m_3}a^kb^{m_2-j}a^{m_1-i-k},$
$a^{m_1}b^{m_2}a^{m_1}c^{m_3}a^{m_1}b^{m_2}a^{m_1},\ a^ib^ja^kc^{m_3}a^db^{m_2-j}a^{m_1-i-k-d};$

where $1 \leq i, k, d \leq m_1 - 1,\ 1 \leq j \leq m_2 - 1$.

This establishes the result of this lemma. ■

Corollary 3.4.2 *If I is the set of idempotents in $\overline{\mathcal{S}}_3$, then*

$$|I| = m_1^2(m_1m_2 + 3m_2 + 1) + 3m_1m_2 + 2m_2 + 4m_1 + 7.$$

In the next section we study the maximal subgroups of $\overline{\mathcal{S}}_2$ and $\overline{\mathcal{S}}_3$.

3.5 The maximal subgroups of $\overline{\mathcal{S}}_2$ and $\overline{\mathcal{S}}_3$

In 1.5 we have seen that the maximal subgroups of a semigroup $\mathcal{S}$ coincide with the $\mathcal{H}$-classes of $\mathcal{S}$ which contain idempotents. In this section we will find the maximal subgroups of $\overline{\mathcal{S}}_2$ and $\overline{\mathcal{S}}_3$.

Since each subgroup of any semigroup contains one and only one idempotent, we take each idempotent form in Lemma 3.4.2(Lemma 3.4.3) and build up the maximal subgroup containing that idempotent using Lemma 3.2.1 (Lemma 3.3.1) and the other properties that we have studied in the previous sections. The next lemmas help in finding the maximal subgroups of $\overline{\mathcal{S}}_n$ in general.

Lemma 3.5.1 *Let $u, v \in N(\overline{\mathcal{S}}_{k-1}) \cup (\overline{\mathcal{S}}_k^1 \setminus \overline{\mathcal{S}}_k)$ such that $w = ua_k^{m_k}v$ is an idempotent, let G_w be the maximal subgroup containing w and let $g \in G_w$. Then, $g = xa_k^iy$ for some $x, y \in N(\overline{\mathcal{S}}_{k-1}) \cup (\overline{\mathcal{S}}_k^1 \setminus \overline{\mathcal{S}}_k)$ and $1 \leq i \leq m_k$.*

Proof. Let $g \in G_w$. Since w is an idempotent, it follows that

$$g = gw = gua_k^{m_k}v \in \overline{\mathcal{S}}_k \setminus \overline{\mathcal{S}}_{k-1}.$$

It then follows by Theorem 3.3.1 that $g = xa_k^i y$ for some $x, y \in N(\overline{\mathcal{S}}_{k-1}) \cup (\overline{\mathcal{S}}_k^1 \setminus \overline{\mathcal{S}}_k)$ and $1 \leq i \leq m_k$. ■

Lemma 3.5.2 *Let $x, y, u, v \in N(\overline{\mathcal{S}}_{k-1}) \cup (\overline{\mathcal{S}}_k^1 \setminus \overline{\mathcal{S}}_k)$, and let $w = ua_k^{m_k} v$ be an idempotent in $\overline{\mathcal{S}}_k$. Then*

$$(xa_k^i y)w = w(xa_k^i y) = xa_k^i y \tag{3.24}$$

if and only if $y = v$ and $xvu = x = uvx$.

Proof. Assume that (3.24) holds. Then, Lemma 3.3.3 implies that

$$xa_k^i y = xa_k^i yw = xa_k^i yua_k^{m_k} v = xyua_k^i v,$$

it then follows from Theorem 3.3.1 that $xyu = x$ and $y = v$. Hence, $xvu = x$. Similarly,

$$xa_k^i y = wxa_k^i y = uvxa_k^i y$$

which implies that $uvx = x$. Conversely, if $xvu = uvx = x$ and $y = v$, then (3.24) holds. ■

Theorem 3.5.1 *The maximal subgroups of $\overline{\mathcal{S}}_2$ are (up to isomorphism):*

1. $\langle\ x\ :\ x^{m_1} = 1\ \rangle$, $\langle\ x : x^{m_2} = 1\ \rangle$,
2. $\langle\ x,\ y\ :\ x^{m_1} = y^{m_2} = xyx^{-1}y^{-1} = 1\ \rangle$.

Proof. Let G_x stands for the maximal subgroup of $\overline{\mathcal{S}}_2$ containing the idempotent x (H_x when considered as an $\mathcal{H}$-class). Consider $G_{a^{m_1}}$. It follows from Lemma 3.5.1 that if $w \in G_{a^{m_1}}$, then

$$w = a^i : 1 \leq i \leq m_1.$$

Hence,

$$G_{a^{m_1}} = \{ a^i : 1 \leq i \leq m_1 \} \cong G(\langle x : x^{m_1} = 1 \rangle).$$

Similarly, Lemma 3.5.1 and Lemma 3.5.2 imply that

$$G_{b^{m_2}} \cong G(\langle x : x^{m_2} = 1 \rangle).$$

Thus $G_{a^{m_1}}$ and $G_{b^{m_2}}$ are of type 1.

On the other hand if $w \in G_{a^{m_1}b^{m_2}}$, then Lemma 3.5.1 and Lemma 3.5.2 imply that

$$w = xb^j : x \in \{a^i : 1 \leq i \leq m_1\}, 1 \leq j \leq m_2.$$

Thus $G_{a^{m_1}b^{m_2}} \subseteq \{a^ib^j : 1 \leq i \leq m_1, 1 \leq j \leq m_2\}$. Since if $w = a^ib^j$ and $w' = a^{m_1-i}b^{m_2-j}$ then $ww' = w'w = a^{m_1}b^{m_2}$, it follows that

$$\{a^ib^j : 1 \leq i \leq m_1, 1 \leq j \leq m_2\}$$

is a group. Since $G_{a^{m_1}b^{m_2}}$ is a maximal, it follows that

$$G_{a^{m_1}b^{m_2}} = \{ a^ib^j : 1 \leq i, j \leq m_1, m_2 \}.$$

To show that $G_{a^{m_1}b^{m_2}}$ is of type 2, let G be the semigroup defined by:

$$P = \langle x, y : x^{m_1} = y^{m_2} = xyx^{-1}y^{-1} = 1 \rangle,$$

and let $x_0 = ab^{m_2}$, $y_0 = a^{m_1}b$. Then

$$x_0^iy_0^j = a^ib^j \text{ (Lemma 3.2.1)}.$$

Thus, x_0 and y_0 generate $G_{a^{m_1}b^{m_2}}$ (i.e. $G_{a^{m_1}b^{m_2}} \subseteq \langle x_0, y_0 \rangle$). Since x_0 and y_0 are in $G_{a^{m_1}b^{m_2}}$, $\langle x_0, y_0 \rangle \subseteq G_{a^{m_1}b^{m_2}}$ and thus $G_{a^{m_1}b^{m_2}} = \langle x_0, y_0 \rangle$. Also, x_0 and y_0 satisfy the relations in P and thus $\langle x_0, y_0 \rangle$ is a homomorphic image of G (Proposition 3.2.1). In fact the mapping $\phi : G \longrightarrow \langle x_0, y_0 \rangle$ defined by $(x^iy^j)\phi = x_0^iy_0^j$, where $0 \le i,j \le m_1, m_2$ respectively, is a homomorphism onto $\langle\ x_0,\ y_0\ \rangle$ (note that ϕ is the homomorphic extension of $f : \{x,y\} \longrightarrow \langle x_0, y_0\rangle$ which takes $x \mapsto x_0$ and $y \mapsto y_0$). Furthermore, ϕ is injective because if $g_1, g_2 \in G$ such that $(g_1)\phi = (g_2)\phi$ then

$$g_1 = x^iy^j, g_2 = x^dy^r \Longrightarrow x_0^iy_0^j = x_0^dy_0^r \Longrightarrow a^ib^j = a^db^r \text{ (Lemma 3.2.1)}$$

$$\Longrightarrow i = d, j = r \Longrightarrow g_1 = g_2 \text{ (Theorem 3.2.1)}$$

and thus $G_{a^{m_1}b^{m_2}} \cong G$. Similarly, one can show that

$$G_{a^ib^{m_2}a^{m_1-i}} \ \cong\ G(\langle x,\ y : x^{m_1} = y^{m_2} = xyx^{-1}y^{-1} = 1\ \rangle)$$

where $x_0 = ab^{m_2}a^{m_2-i},\ y_0 = a^{m_1}ba^{m_1-i}$. ■

We summarize the details of the maximal subgroups of $\overline{S}_2$ in the following table:

maximal subgroup	presentation	generators
$G_{a^{m_1}}$	(1)	$x = a$
$G_{a^{m_2}}$	(1)	$x = b$
$G_{a^{m_1}b^{m_2}}$	(2)	$x = ab^{m_2},\ y = a^{m_1}b$
$G_{a^ib^{m_2}a^{m_1-i}} : 1 \le i \le m_1$	(2)	$x = a^{i+1}b^{m_2}a^{m_1-i},\ y = a^iba^{m_1-i}$

Table 3.6:The maximal subgroups of $\overline{S}_2$.

The next theorem gives the maximal subgroups of $\overline{S}_3$.

Theorem 3.5.2 *The maximal subgroups of $\overline{S}_3$ are (up to isomorphism):*

1. $\langle\ x\ :\ x^{m_1} = 1\ \rangle$,
2. $\langle\ x\ :\ x^{m_2} = 1\ \rangle$,
3. $\langle\ x\ :\ x^{m_3} = 1\ \rangle$,
4. $\langle\ x,\ y\ :\ x^{m_1} = y^{m_2} = xyx^{-1}y^{-1} = 1\ \rangle$,

5. $\langle\ x,\ y\ :\ x^{m_1} = y^{m_3} = xyx^{-1}y^{-1} = 1\ \rangle$,

6. $\langle\ x,\ y\ :\ x^{m_2} = y^{m_3} = xyx^{-1}y^{-1} = 1\ \rangle$,

7. $\langle\ x,\ y,\ z:\ x^{m_1} = y^{m_2} = z^{m_3} = xyx^{-1}y^{-1} = xzx^{-1}z^{-1} = yzy^{-1}z^{-1} = 1\ \rangle$.

Proof. We show that the maximal subgroup containing the idempotent

$$w = a^i b^j a^k c^{m_3} a^d b^{m_2-j} a^{m_1-i-k-d}$$

is of type 7 and the others can be proved in a similar way. So, let

$$u = a^i b^j a^k,\ v = a^d b^{m_2-j} a^{m_1-i-k-d},$$

and let G_w be the maximal subgroup containing w. Assume that

$$z = xc^t s \in G_w : x, s \in N(\overline{\mathcal{S}}_2) \cup (\overline{\mathcal{S}}_3^1 \setminus \overline{\mathcal{S}}_3), 1 \leq t \leq m_3.$$

It then follows from Lemma 3.5.1 that $s = v$ and $xvu = x = uvx$. Hence, Lemma 3.2.1 implies that

$$uv = a^i b^j a^k a^d b^{m_2-j} a^{m_1-i-k-d} = a^{i+k+d} b^{m_2} a^{m_1-i-k-d}, \tag{3.25}$$

$$vu = a^d b^{m_2-j} a^{m_1-i-k-d} a^i b^j a^k = a^{m_1-k} b^{m_2} a^k. \tag{3.26}$$

The combination of (3.25) and (3.26) implies, in order to have $xvu = x$ and $uvx = x$, that x is of the form $a^{i'} b^{j'} a^{k'}$ for some $i', j', k' \in \mathbf{N}$; and (3.26) implies that $k' = k$. Thus, z is of the form $a^{i'} b^{j'} a^k c^t v$. Since each element in the form $a^{i'} b^{j'} a^k c^t v$ has $a^{2i-(i'+k'+d)} b^{2j-j'} a^k c^{m_3-t} v$ as its inverse with respect to the identity w, we must have

$$G_w = \{a^{i'} b^{j'} a^k c^t v : 1 \leq i' \leq m_1, 1 \leq j' \leq m_2, 1 \leq t \leq m_3\}$$

To show that G_w is of type 7, let

$$\begin{aligned}
x &= a^{i+1}b^j a^k c^{m_3} a^d b^{m_2-j} a^{m_1-(i+k+d)},\\
y &= a^i b^{j+1} a^k c^{m_3} a^d b^{m_2-j} a^{m_1-(i+k+d)},\\
z &= a^i b^j a^k c a^d b^{m_2-j} a^{m_1-(i+k+d)}.
\end{aligned}$$

Using Lemma 3.3.1 and Corollary 3.3.1, one can easily show that

$$\begin{aligned}
x^r &= a^{r+i}b^j a^k c^{m_3} a^d b^{m_2-j} a^{m_1-(i+k+d)},\\
y^r &= a^i b^{r+j} a^k c^{m_3} a^d b^{m_2-j} a^{m_1-(i+k+d)},\\
z^r &= a^i b^j a^k c^r a^d b^{m_2-j} a^{m_1-(i+k+d)}.
\end{aligned}$$

Therefore,

$$x^{m_1} = y^{m_2} = z^{m_3} = a^i b^j a^k c^{m_3} a^d b^{m_2-j} a^{m_1-(i+k+d)} = w,$$

$$x^p y^q z^t = a^{p+i} b^{q+j} a^k c^t a^d b^{m_2-j} a^{m_1-(i+k+d)}.$$

Hence,

$$a^{i'} b^{j'} a^k c^t a^d b^{m_2-j} a^{m_1-(i+k+d)} = x^{i'-i} y^{j'-j} z^t.$$

This shows that x, y, z generate G_w. Also, Lemma 3.3.1 and Corollary 3.3.1 shows that x, y and z satisfy the relations in 7 and thus G_w is a homomorphic image of the group defined by the presentation in (7) which is $C_{m_1} \times C_{m_2} \times C_{m_3}$. Since $|G_w| = m_1 m_2 m_3 = |C_{m_1} \times C_{m_2} \times C_{m_3}|$, it follows that $G_w \cong C_{m_1} \times C_{m_2} \times C_{m_3}$ and thus G_w is of type 7. ■

We summarize the details of the maximal subgroups of $\overline{\mathcal{S}}_3$ in the following table:

No	maximal subgroup	P	generators
1	$G_{a^{m_1}}$	(1)	$x = a$
2	$G_{b^{m_2}}$	(2)	$x = b$
3	$G_{c^{m_3}}$	(3)	$x = c$
4	$G_{a^{m_1}b^{m_2}}$	(4)	$x = ab^{m_2},\ y = a^{m_1}b$
5	$G_{a^{m_1}c^{m_3}}$	(5)	$x = ac^{m_3},\ y = a^{m_1}c$
6	$G_{b^{m_2}c^{m_3}}$	(6)	$x = bc^{m_3},\ y = b^{m_2}c$
7	$G_{a^ib^{m_2}a^{m_1-i}}$	(4)	$x = a^{i+1}b^{m_2}a^{m_1-i},\ y = a^iba^{m_1-i}$
8	$G_{a^ib^{m_2}a^{m_1-i}c^{m_3}}$	(7)	$x = a^{i+1}b^{m_2}a^{m_1-i}c^{m_3},\ y = a^iba^{m_1-i}c^{m_3},$ $z = a^ib^{m_2}a^{m_1-i}c$
9	$G_{a^ib^{m_2}a^kc^{m_3}a^{m_1-(i+k)}}$	(7)	$x = a^{i+1}b^{m_2}a^kc^{m_3}a^{m_1-(i+k)},\ y = a^iba^kc^{m_3}a^{m_1-(i+k)},$ $z = a^ib^{m_2}a^kca^{m_1-(i+k)}$
10	$G_{a^ib^ja^{m_1-i}c^{m_3}b^{m_2-j}}$	(7)	$x = a^{i+1}b^{m_2}a^{m_1-i}c^{m_3}b^{m_2-j},\ y = a^iba^{m_1-i}c^{m_3}b^{m_2-j},$ $z = a^ib^{m_2}a^{m_1-i}cb^{m_2-j}$
11	$G_{a^ib^ja^kc^{m_3}a^{m_1-(i+k)}b^{m_2-j}}$	(7)	$x = a^{i+1}b^{m_2}a^kc^{m_3}a^{m_1-(i+k)}b^{m_2-j},\ y = a^iba^kc^{m_3}$ $a^{m_1-(i+k)}b^{m_2-j},\ \ z = a^ib^{m_2}a^kca^{m_1-(i+k)}b^{m_2-j}$
12	$G_{a^ib^ja^kc^{m_3}b^{m_2-j}a^{m_1-(i+k)}}$	(7)	$x = a^{i+1}b^{m_2}a^kc^{m_3}b^{m_2-j}a^{m_1-(i+k)},\ y = a^iba^kc^{m_3}$ $b^{m_2-j}a^{m_1-(i+k)},\ \ z = a^ib^{m_2}a^kcb^{m_2-j}a^{m_1-(i+k)}$
13	$G_{a^ib^ja^kc^{m_3}a^db^{m_2-j}a^{m_1-(i+k+d)}}$	(7)	$x = a^{i+1}b^{m_2}a^kc^{m_3}a^db^{m_2-j}a^{m_1-(i+k+d)},$ $y = a^iba^kc^{m_3}a^db^{m_2-j}a^{m_1-(i+k+d)},$ $z = a^ib^{m_2}a^kca^db^{m_2-j}a^{m_1-(i+k+d)}$
14	$G_{a^ic^{m_3}a^{m_1-i}}$	(5)	$x = a^{i+1}c^{m_3}a^{m_1-i},\ y = a^ica^{m_1-i}$
15	$G_{b^jc^{m_3}b^{m_2-j}}$	(6)	$x = b^{j+1}c^{m_3}b^{m_2-j},\ y = b^jcb^{m_2-j}$
16	$G_{a^{m_1}b^{m_2}c^{m_3}}$	(7)	$x = ab^{m_2}c^{m_3},\ y = a^{m_1}bc^{m_3},\ z = a^{m_1}b^{m_2}c$
17	$G_{a^{m_1}b^jc^{m_3}b^{m_2-j}}$	(7)	$x = ab^jc^{m_3}b^{m_2-j},\ y = a^{m_1}b^{j+1}c^{m_3}b^{m_2-j},$ $z = a^{m_1}b^jcb^{m_2-j}$
18	$G_{a^ib^{m_2}c^{m_3}a^{m_1-i}}$	(7)	$x = a^{i+1}b^{m_2}c^{m_3}a^{m_1-i},\ y = a^ibc^{m_3}a^{m_1-i},$ $z = a^ib^{m_2}ca^{m_1-i}$
19	$G_{a^ib^jc^{m_3}a^{m_1-i}b^{m_2-j}}$	(7)	$x = a^{i+1}b^jc^{m_3}a^{m_1-i}b^{m_2-j},\ y = a^ib^{j+1}c^{m_3}a^{m_1-i}b^{m_2-j},$ $z = a^ib^jca^{m_1-i}b^{m_2-j}$
20	$G_{a^ib^jc^{m_3}b^{m_2-j}}a^{m_1-i}$	(7)	$x = a^{i+1}b^jc^{m_3}b^{m_2-j}a^{m_1-i},\ y = a^ib^{j+1}c^{m_3}b^{m_2-j}a^{m_1-i},$ $z = a^ib^jcb^{m_2-j}a^{m_1-i}$
21	$G_{a^ib^{m_2}c^{m_3}}a^{m_1-i}$	(7)	$x = a^{i+1}b^{m_2}c^{m_3}a^{m_1-i},\ y = a^ibc^{m_3}a^{m_1-i},$ $z = a^ib^{m_2}ca^{m_1-i}$

Table 3.7:The maximal subgroups in $\overline{S}_3$.

Chapter 4

Semigroups related to dihedral groups

The group presentation

$$\langle\ a, b : a^2 = b^m = (ab)^2 = e\ \rangle \tag{4.1}$$

defines the *dihedral* group D_m of order $2m$. E. F. Robertson and Y. Ünlü [44] examined the semigroup presentation

$$\langle\ a, b : a^3 = a, b^{m+1} = b, bab = a\ \rangle$$

which, when considered as a group presentation, defines D_m.

Consider the following presentation:

$$P = \langle a, b, c : a^{\ell+1} = a, b^{m+1} = b, c^{n+1} = c, a^{\ell}ba = ab^{m-1}, a^{\ell}ca = ac^{n-1}, bc = cbc^{n}\rangle$$

where $\ell, m, n \geq 2$.

When $\ell = 2$ and $m = n$, $G(P)$ is the square product $D_m \times D_m$ of the *dihedral* group D_m.

In this chapter we carry on using the technique we used in Chapter 3 to study the semigroup defined by P which we denote by $\mathcal{S}_{(\ell,m,n)}$. We first consider the case $\ell = 2$ and $n = m$; we denote the semigroup defined by P in this case by $\mathcal{S}_m$. In the first section we give a normal form for $\mathcal{S}_m$ and a formula for its order. In Section 2 we find the set of idempotents of $\mathcal{S}_m$ and in Section 3 we develop a new method to find the maximal subgroups of a semigroup $\mathcal{S}$ and then we apply that method to find the maximal subgroups of $\mathcal{S}_m$. In Section 4 we consider the general case $\mathcal{S}_{(\ell,m,n)}$.

4.1 A normal form for $\mathcal{S}_m$

As we have seen in Chapter 3, normal forms are good tools to study semigroups. So we devote this section to finding a normal form for the semigroup $\mathcal{S}_m$ which is defined by

$$P = \langle a, b, c : a^3 = a, b^{m+1} = b, c^{m+1} = c, a^2ba = ab^{m-1}, a^2ca = ac^{m-1}, bc = cbc^m \rangle.$$

Remark 4.1.1 In this chapter, the non-zero powers of a are reduced modulo 2 and the non-zero powers of b and c are reduced modulo m, unless otherwise stated. Also, $x \pmod m$ $[x \pmod 2]$ is in the range $1, \ldots, m$ $[1, 2]$ for all $x \in \mathbf{N}$; and when $i = m$, $x^{m-i} = x^m$ for all $x \in \{a, b, c\}$, unless otherwise stated.

Lemma 4.1.1 *Let a, b and c be as in the presentation of $\mathcal{S}_m$, and let $1 \leq i, j \leq m$. Then,*

1. $ab^ia^2 = ab^i$,
2. $ac^ia^2 = ac^i$,
3. $ab^ia = a^2b^{m-i}$,
4. $ac^ia = a^2c^{m-i}$,
5. $ab^ic^ja^2 = ab^ic^j$,

6. $ac^jb^ia^2 = ac^jb^i$,

7. $ab^ic^ja = a^2b^{m-i}c^{m-j}$,

8. $ac^jb^ia = a^2c^{m-j}b^{m-i}$,

9. $cb^ic = b^ic^2$;

Proof. (1) Since $(b^{m-1})^{m-i} = b^{m(m-i-1)+i} = b^i$, it is enough to show that

$$a(b^{m-1})^ka^2 = a(b^{m-1})^k. \tag{4.2}$$

We use mathematical induction. When $k = 1$, we get

$$ab^{m-1}a^2 = a^2ba.a^2 = a^2ba = ab^{m-1}.$$

This shows that (4.2) is true when $k = 1$.

Assume (4.2) is true for all $k \leq t$ for some $t < m$. Then,

$$a(b^{m-1})^{t+1}a^2 = a(b^{m-1})^t(b^{m-1})a^2 = a(b^{m-1})^ta^2(b^{m-1})a^2$$

$$= a(b^{m-1})^ta^2b^{m-1} = a(b^{m-1})^{t+1}.$$

Hence, (4.2) is true for all $k : 1 \leq k \leq m$. It follows that

$$ab^ia^2 = a(b^{m-1})^{m-i}a^2 = a(b^{m-1})^{m-i} = ab^i.$$

(2) The proof follows as in (1).

(3) When $i = 1$, we get

$$aba = a(a^2ba) = a(ab^{m-1}) = a^2b^{m-1}.$$

This shows that $ab^ia = ab^{m-i}$ when $i = 1$.

Assume that $ab^ia = ab^{m-i}$ for all positive integers $i : 1 \leq i \leq k$ for some $k < m$. Part (1) of this lemma implies that,

$$ab^{k+1}a = \underbrace{ab^k}\, ba = \underbrace{ab^k(a^2}\, ba) = ab^k\ (ab^{m-1})$$

$$= a^2b^{m-k}.b^{m-1} = a^2b^{m-(k+1)}.$$

Hence, $ab^ia = ab^{m-i}$ for all $1 \leq i \leq m$.

(4) The proof follows as in (3).

(5) Part (1) implies that

$$ab^ic^ja^2 = ab^ia^2c^ja^2 = ab^ia^2c^j = ab^ic^j \quad \text{(by (2))}.$$

(6) The proof follows as in (5).

(7) Parts (1), (4) and (3) of this lemma imply that

$$ab^ic^ja = ab^ia^2c^ja = ab^iac^{m-j} = a^2b^{m-i}c^{m-j}$$

(8) The proof follows as in (7).

(9) When $i = 1, cbc = cbc^m.c = bc.c = bc^2$ i.e. it is true for $i = 1$.

Assume it is true for all $i : 1 \leq i \leq k$ for some $k < m$. Then,

$$cb^{k+1}c = cb^k(bc) = cb^k(cbc^m) = b^kc^2bc^m = b^{k+1}c^2.$$

This establishes the lemma. ∎

Corollary 4.1.1 *Let a, b and c be as in the presentation of $\mathcal{S}_m$, and let $1 \leq i, j, k \leq m$. Then,*

$$c^ib^jc^k = b^jc^{i+k}.$$

Theorem 4.1.1 *Let $N(\mathcal{S}_m)$ be the subset of $\{a, b, c\}^+$ defined by:*

$$N(\mathcal{S}_m) = \{b^j, b^ic^jb^k, b^ic^jb^ka^lb^dc^rb^t, b^ia^lb^dc^rb^t, b^ic^jb^ka^lb^d, b^ia^lb^k : \\ 0 \leq i,k,d,t \leq m, 1 \leq j,r \leq m, 1 \leq l \leq 2\}.$$

Then for any $w \in \{a,b,c\}^+$ there exists $w^* \in N(\mathcal{S}_m)$ such that $w = w^*$ in $\mathcal{S}_m$.

Proof. Corollary 4.1.1 implies that every element of $\{b,c\}^+$ is equal to a word of the form $b^ic^jb^k : 0 \leq i,j,k \leq m$ in $\mathcal{S}_m$. Also Lemma 4.1.1 and Corollary 4.1.1 imply that any element of $\{a,b,c\}^+ \setminus \{b,c\}^+$ is equal to a word of the form $b^ic^jb^ka^qb^dc^rb^t$ where $0 \leq i,j,k,d,r,t \leq m; 1 \leq q \leq 2$ in $\mathcal{S}_m$. ∎

In order to show that the words in $N(\mathcal{S}_m)$ represent distinct elements in $\mathcal{S}_m$ we will define three mappings α, β and γ such that the sets $A = \{a,b,c\}$ and $B = \{\alpha, \beta, \gamma\}$ satisfy Proposition 3.2.1.

Let $p_1, p_2, \ldots, p_{11}$ be any distinct primes greater than 11. Define $\overline{A}$, $\overline{B}$, C, D, E, F, G and X as follows:

- $\overline{A} = \{2^i.3^jp_1 : 0 \leq i,j \leq m\}$,
- $\overline{B} = \{2^i.3^jp_2 : 0 \leq i,j \leq m\}$,
- $C = \{5^i.7^jp_3 : 0 \leq i,j \leq m\}$,
- $D = \{2^ip_4 : 0 \leq i \leq m\}$,
- $E = \{p_5, p_6, p_7\}$,
- $F = \{11^ip_8 : 0 \leq i \leq 2\}$,
- $G = \{p_9, p_{10}, p_{11}\}$,
- $X = \overline{A} \cup \overline{B} \cup C \cup D \cup E \cup F \cup G$.

Define $\alpha, \beta, \gamma : X \to X$ as follows:

$$(x)\alpha = \begin{cases} x : x \in \{p_1, p_3, p_4\}, \\ p_3 : x \in \overline{B}, \\ 2^{m-i}p_1 : x = 2^i p_1, 1 \leq i \leq m, \\ 3^{m-j}p_1 : x = 3^j p_1, 1 \leq j \leq m, \\ 2^{m-i}.3^{m-j}p_1 : x = 2^i.3^j p_1, 1 \leq i, j \leq m, \\ 5^{m-i}p_3 : x = 5^i p_3, 1 \leq i \leq m, \\ 7^{m-j}p_3 : x = 7^j p_3, 1 \leq j \leq m, \\ 5^{m-i}.7^{m-j}p_3 : x = 5^i.7^j p_3, 1 \leq i, j \leq m, \\ 2^{m-i}p_4 : x = 2^i p_4, 1 \leq i \leq m, \\ p_{(i+1)} : p_7 \neq x \neq p_{11}, x \in E \cup G, \\ p_6 : x = p_7, \\ p_{10} : x = p_{11}. \\ 11x : x \in F, \end{cases}$$

$$(x)\beta = \begin{cases} 2x : x \in \overline{A} \cup \overline{B} \cup D, \\ 5x : x \in C, \\ p_6 : x = p_5, \\ x : x \in (E \setminus \{p_5\}) \cup F \cup G, \end{cases}$$

$$(x)\gamma = \begin{cases} 3x : x \in \overline{A} \cup \overline{B}, \\ 7x : x \in C, \\ p_4 : x \in D, \\ x : x \in E \cup F, \\ p_{10} : x = p_9, \\ x : x \in (G \setminus \{p_9\}). \end{cases}$$

where the non-zero powers of 2,3,5, and 7 are reduced modulo m and the non-zero powers of 11 are reduced modulo 2. Also, we agree that $x^{m-i} = x^m$ whenever $i = m$ and $x \in \{b, c\}$.

The next lemma shows that these three mappings α, β and γ satisfy the relations in the presentation of $\mathcal{S}_m$.

Lemma 4.1.2 *The mappings* α, β, γ *satisfy the following relations:*

$$(1) \quad \alpha^3 = \alpha, \qquad (4) \quad \alpha^2\beta\alpha = \alpha\beta^{m-1},$$

(2) $\beta^{m+1} = \beta$, (5) $\alpha^2\gamma\alpha = \alpha\gamma^{m-1}$,

(3) $\gamma^{m+1} = \gamma$, (6) $\beta\gamma = \gamma\beta\gamma^m$.

Proof. We prove this lemma by showing that the restriction of these three mappings to any of the sets $\overline{A}, \overline{B}, C, D, E, F$ or G satisfy the relations in this lemma. First, recall that in Remark 4.1.1, we agreed to have

$$x^{m-i} = x^m \text{ whenever } i = m.$$

(i) If $x \in \overline{A}$, then $x = 2^i.3^j p_1 : 0 \leq i, j \leq m$. We prove the lemma for the case $x = 2^i.3^j p_1 : 1 \leq i, j \leq m$ and the other cases can be proved in a similar way.

(1) it follows immediately from the definition of α on $\overline{A}$, that for each $x \in \overline{A}$, $(x)\alpha^3 = (x)\alpha$,

(2) $(x)\beta^{m+1} = 2^{(m+1)}x = 2x = (x)\beta$,

(3) $(x)\gamma^{m+1} = 3^{(m+1)}x = 3x = (x)\gamma$,

(4) $(x)\alpha^2\beta\alpha = (x)\beta\alpha = (2x)\alpha = (2^{i+1}.3^j p_1)\alpha = 2^{m-i-1}.3^{m-j}p_1$,

$$(x)\alpha\beta^{m-1} = (2^{m-i}.3^{m-j}p_1)\beta^{m-1} = 2^{m-i}.3^{m-j}.2^{m-1}p_1$$

$$= 2^{m-i-1}.3^{m-j}p_1 = (x)\alpha^2\beta\alpha \text{ (see Remark 4.1.1)},$$

(5) $(x)\alpha^2\gamma\alpha = (x)\gamma\alpha = (2^i.3^{j+1}p_1)\alpha = 2^{m-i}.3^{m-j-1}p_1$,

$$(x)\alpha\gamma^{m-1} = (2^{m-i}.3^{m-j}p_1)\gamma^{m-1} = 2^{m-i}.3^{m-j-1}p_1 = (x)\alpha^2\gamma\alpha$$

(6) $(x)\beta\gamma = 2^{(i+1)}.3^{(j+1)}p_1$,

$$(x)\gamma\beta\gamma^m = 2^{(i+1)}.3^{(j+1)}.3^m p_1$$

$$= 2^{(i+1)}.3^{(j+1)}p_1 = (x)\beta\gamma.$$

(ii) If $x \in \overline{B}$ then $x = 2^i 3^j p_2 : 0 \leq i, j \leq m$. We prove the lemma for the case $x = 2^i.3^j p_2 : 1 \leq i, j \leq m$ and the other cases can be proved in a similar way.

(1) $(x)\alpha = p_3, (x)\alpha^2 = (p_3)\alpha = p_3 = (x)\alpha,$

(2) $(x)\beta^{m+1} = 2^{(m+1)}x = 2x = (x)\beta,$

(3) $(x)\gamma^{m+1} = 3^{(m+1)}x = 3x = (x)\gamma,$

(4) $(x)\alpha^2\beta\alpha = (5^m.7^m p_3)\beta\alpha = (5.7^m p_3)\alpha = 5^{m-1}p_3$

$(x)\alpha\beta^{m-1} = (p_3)\beta^{m-1} = 5^{m-1}p_3 = (x)\alpha^2\beta\alpha,$

(5) $(x)\alpha^2\gamma\alpha = (p_3)\gamma\alpha = (7p_3)\alpha = 7^{m-1}p_3$

$(x)\alpha\gamma^{m-1} = (p_3)\gamma^{m-1} = 7^{m-1}p_3 = (x)\alpha^2\gamma\alpha.$

(6) The proof is as in (6) of (i).

(iii) If $x \in C$ then $x = 5^i.7^j p_3 : 0 \leq i, j \leq m$. We prove the lemma for the case $x = 5^i.7^j p_3 : 1 \leq i, j \leq m$ and the other cases can be proved in a similar way.

(1) $(x)\alpha^2 = x$ implying that $(x)\alpha^3 = (x)\alpha,$

(2) $(x)\beta^{m+1} = 5^{(m+1)}x = 5x = (x)\beta,$

(3) $(x)\gamma^{m+1} = 7^{(m+1)}x = 7x = (x)\gamma,$

(4) $(x)\alpha^2\beta\alpha = (x)\beta\alpha = (5^{(i+1)}.7^j p_3)\alpha = 5^{m-i-1}.7^{m-j}p_3$

$(x)\alpha\beta^{m-1} = (5^{m-i}.7^{m-j}p_3)\beta^{m-1} = 5^{m-i-1}.7^{m-j}p_3 = (x)\alpha^2\beta\alpha,$

(5) The proof is as in (4).

(6) The proof is as in part (6) of (i).

(iv) If $x \in D$ then $x = 2^i p_4 : 0 \leq i \leq m$. If $x = p_4$, then

(1) $(x)\alpha^3 = p_4 = (x)\alpha,$

(2) $(x)\beta^{m+1} = 2^{(m+1)}p_4 = 2p_4 = (x)\beta,$

(3) $(x)\gamma^{m+1} = p_4 = (p_4)\gamma = (x)\gamma,$

(4) $(x)\alpha^2\beta\alpha = (p_4)\beta\alpha = (2p_4)\alpha = 2^{m-1}p_4$

$(x)\alpha\beta^{m-1} = (p_4)\beta^{m-1} = 2^{m-1}p_4 = (x)\alpha^2\beta\alpha,$

(5) $(x)\alpha^2\gamma\alpha = (p_4)\gamma\alpha = (p_4)\alpha = p_4$

$$(x)\alpha\gamma^{m-1} = (p_4)\gamma^{m-1} = p_4 = (x)\alpha^2\gamma\alpha,$$

(6) $(x)\beta\gamma = (2p_4)\gamma = p_4$

$$(x)\gamma\beta\gamma^m = (p_4)\beta\gamma^m = (2p_4)\gamma^m) = p_4 = (x)\beta\gamma;$$

if $x = 2^i p_4 : i \geq 1$ then :

(1) $(x)\alpha^2 = x$ implying that $(x)\alpha^3 = (x)\alpha$,

(2) $(x)\beta^{m+1} = 2^{(m+1)}x = 2x = (x)\beta$,

(3) $(x)\gamma^{m+1} = p_4 = (x)\gamma$,

(4) $(x)\alpha^2\beta\alpha = (x)\beta\alpha = (2^{i+1}p_4)\alpha = 2^{m-i-1}p_4$,

$$(x)\alpha\beta^{m-1} = (2^{m-i}p_4)\beta^{m-1} = 2^{m-i-1}p_4 = (x)\alpha^2\beta\alpha,$$

(5) $(x)\alpha^2\gamma\alpha = (x)\gamma\alpha = (p_4)\alpha = p_4$

$$(x)\alpha\gamma^{m-1} = (2^{m-i}p_4)\gamma^{m-1} = p_4 = (x)\alpha^2\gamma\alpha.$$

(v) It is clear that for $x \in E \cup F \cup G$ all the mentioned relations hold.

■

The following table shows the distinct and related elements in the semigroup generated by α and β, $S(\alpha, \beta)$.

No	w	action on p_1	p_2	p_4	p_5	p_9	related to
1	α^2	p_1	p_3	p_4	p_7	p_{11}	
2	β^j	$2^j p_1$	$2^j p_2$	$2^j p_4$	p_6	p_9	
3	γ^j	$3^j p_1$	$3^j p_2$	p_4	p_5	p_{10}	
4	$\alpha^2\beta^i$	$2^i p_1$	$5^i p_3$	$2^i p_4$	p_7	p_{11}	
5	$\beta^i\alpha^2$	$2^i p_1$	p_3	$2^i p_4$	p_7	p_{11}	
6	$\alpha^2\gamma^j$	$3^j p_1$	$7^j p_3$	p_4	p_7	p_{11}	
7	$\gamma^j\alpha^2$	$3^j p_1$	p_3	p_4	p_7	p_{10}	
8	$\beta^i\gamma^j$	$2^i.3^j p_1$	$2^i.3^j p_2$	p_4	p_6	p_{10}	
9	$\gamma^j\beta^i$	$2^i.3^j p_1$	$2^i.3^j p_2$	$2^i p_4$	p_6	p_{10}	
10	$\alpha^2\beta^i\gamma^j$	$2^i.3^j p_1$	$5^i.7^j p_3$	p_4	p_7	p_{11}	
11	$\alpha^2\beta^d\gamma^r\beta^t$	$2^{d+t}.3^r p_1$	$5^{d+t}.7^r p_3$	$2^t p_4$	p_7	p_{11}	
12	$\alpha^2\gamma^j\beta^i$	$2^i.3^j p_1$	$5^i.7^j p_3$	$2^i p_4$	p_7	p_{11}	11
13	$\beta^i\alpha^2\beta^d$	$2^{i+d} p_1$	$5^d p_3$	$2^{i+d} p_4$	p_6	p_{11}	
14	$\beta^i\alpha^2\beta^d\gamma^j$	$2^{i+d}.3^j p_1$	$5^d.7^j p_3$	p_4	p_6	p_{11}	
15	$\beta^i\alpha^2\beta^d\gamma^j\beta^k$	$2^{i+d+k}.3^j p_1$	$5^{d+k}.7^j p_3$	$2^k p_4$	p_6	p_{11}	
16	$\beta^i\alpha^2\gamma^j$	$2^i.3^j p_1$	$7^j p_3$	p_4	p_6	p_{11}	
17	$\beta^i\alpha^2\gamma^j\beta^d$	$2^{i+d}.3^j p_1$	$5^d.7^j p_3$	$2^i p_4$	p_6	p_{11}	15
18	$\beta^i\gamma^j\alpha^2$	$2^i.3^j p_1$	p_3	p_4	p_6	p_{10}	
19	$\beta^i\gamma^j\alpha^2\beta^d$	$2^{i+d}.3^j p_1$	$5^d p_3$	$2^d p_4$	p_6	p_{10}	
20	$\beta^i\gamma^j\alpha^2\gamma^r$	$2^i.3^{j+r} p_1$	$7^r p_3$	p_4	p_6	p_{10}	
21	$\beta^i\gamma^j\alpha^2\beta^d\gamma^r$	$2^{i+d}.3^{j+r} p_1$	$5^d.7^r p_3$	p_4	p_6	p_{10}	10
22	$\beta^i\gamma^j\alpha^2\gamma^r\beta^d$	$2^{i+d}.3^{j+r} p_1$	$5^d.7^r p_3$	$2^d p_4$	p_6	p_{10}	
23	$\beta^i\gamma^j\alpha^2\beta^d\gamma^r\beta^k$	$2^{i+d+k}.3^{j+r} p_1$	$5^{d+k}.7^r p_3$	$2^k p_4$	p_6	p_{10}	22
24	$\beta^i\gamma^j\beta^k$	$2^{i+k}.3^j p_1$	$2^{i+k}.3^j p_2$	$2^k p_4$	p_6	p_{10}	9
25	$\beta^d\gamma^r\beta^t\alpha^2$	$2^{d+t}.3^r.p_1$	p_3	$2^t p_4$	p_6	p_{10}	18
26	$\beta^i\gamma^j\beta^k\alpha^2\beta^d$	$2^{i+k+d}.3^j p_1$	$5^d p_3$	$2^{k+d} p_4$	p_6	p_{10}	19
27	$\beta^d\gamma^j\beta^t\alpha^2\gamma^r$	$2^{d+t}.3^{j+r}.p_1$	$7^r p_3$	p_4	p_6	p_{10}	20
28	$\beta^i\gamma^j\beta^k\alpha^2\beta^d\gamma^r$	$2^{i+k+d}.3^{j+r} p_1$	$5^d.7^r p_3$	p_4	p_6	p_{10}	21
29	$\beta^i\gamma^j\beta^d\alpha^2\gamma^r\beta^k$	$2^{i+d+k}.3^{j+r}.p_1$	$5^k.7^r p_3$	$2^k p_4$	p_6	p_{10}	22
30	$\beta^i\gamma^j\beta^k\alpha^2\beta^d\gamma^r\beta^t$	$2^{i+k+d+t}.3^{j+r} p_1$	$5^{d+t}.7^j p_3$	$2^t p_4$	p_6	p_{10}	22
31	$\gamma^j\alpha^2\beta^d$	$2^d.3^j p_1$	$5^d p_3$	$2^d p_4$	p_7	p_{10}	
32	$\gamma^j\alpha^2\beta^d\gamma^r$	$2^d.3^{j+r} p_1$	$5^d.7^r p_3$	p_4	p_7	p_{10}	
33	$\gamma^j\alpha^2\beta^d\gamma^r\beta^k$	$2^{d+k}.3^{j+r} p_1$	$5^{d+k}.7^r p_3$	$2^k p_4$	p_7	p_{10}	
34	$\gamma^j\alpha^2\gamma^r$	$3^{j+r} p_1$	$7^r p_3$	p_4	p_7	p_{10}	
35	$\gamma^j\alpha^2\gamma^r\beta^i$	$2^i.3^{j+r} p_1$	$5^i.7^r p_3$	$2^i p_4$	p_7	p_{10}	33
36	$\gamma^j\beta^i\alpha^2$	$2^i.3^j p_1$	p_3	$2^i p_4$	p_6	p_{10}	18
37	$\gamma^j\beta^i\alpha^2\beta^d$	$2^{i+d}.3^j p_1$	$5^d p_3$	$2^{i+d} p_4$	p_6	p_{10}	19
38	$\gamma^j\beta^i\alpha^2\gamma^r$	$2^i.3^{j+r} p_1$	$7^r p_3$	p_4	p_6	p_{10}	20
39	$\gamma^j\beta^i\alpha^2\beta^d\gamma^r$	$2^{i+d}.3^{j+r} p_1$	$5^d.7^r p_3$	p_4	p_6	p_{10}	10
40	$\gamma^j\beta^i\alpha^2\gamma^r\beta^d$	$2^{i+d}.3^{j+r} p_1$	$5^d.7^r p_3$	$2^d p_4$	p_6	p_{10}	22
41	$\gamma^j\beta^k\alpha^2\beta^d\gamma^r\beta^t$	$2^{k+d+t}.3^{j+r} p_1$	$5^{d+t}.7^r p_3$	$2^t p_4$	p_6	p_{10}	22

Table 4.1: The action of α, β and γ on p_1, p_2, p_4, p_5 and p_9.

The next lemma shows that replacing α, β and γ by a, b and c in the related words in table 4.1 gives words that represent distinct element in $\mathcal{S}_m$.

Lemma 4.1.3 *Let a, b and c be as in the presentation of $\mathcal{S}_m$, and let $1 \leq i, j, k, d, r, t, i', j', k', d', r', t', i_s, j_s, d_s, k_s, r_s, t_s \leq m$, $1 \leq s \leq 6, 1 \leq q, q' \leq 2$. Then,*

1. $c^j b^i \neq b^d c^r b^k$,
2. $a^q c^j b^i \neq a^{q'} b^d c^r b^k$,
3. $b^i c^j a^q, b^{i'} c^{j'} b^d a^{q'}, c^r b^k a^s$ *represent distinct elements in* $\mathcal{S}_m$,
4. $b^i a^q b^d c^j b^k \neq b^{i'} a^{q'} c^{j'} b^{k'}$,
5. $b^i c^j a^q b^d \neq b^{i'} c^{j'} b^{d'} a^{q'} b^k$,
6. $b^i c^j a^q c^r \neq b^{i'} c^{j'} b^d a^{q'} c^{r'}$,
7. $b^i c^j a^q b^d c^r \neq b^{i'} c^{j'} b^{d'} a^{q'} b^k c^{r'}$,
8. $c^j a^q b^i c^r b^d \neq c^{j'} a^{q'} c^{r'} b^{i'}$,
9. $a^q b^i c^j, c^{j'} b^d a^{q'} b^k c^r, b^{i'} c^{r'} a^s b^{d'} c^t$ *represent distinct elements in* $\mathcal{S}_m$,
10. $b^{i_1} c^{j_1} a^{q_1} c^{r_1} b^{t_1}, b^{i_2} c^{j_2} a^{q_2} b^{k_2} c^{r_2} b^{t_2}, b^{i_3} c^{j_3} b^{d_3} a^{q_3} c^{r_3} b^{t_3}, b^{i_4} c^{j_4} b^{d_4} a^{q_4} b^{k_4} c^{r_4} b^{t_4}$, $c^{j_5} b^{d_5} a^{q_5} c^{r_5} b^{t_5}, c^{j_6} b^{d_6} a^{q_6} b^{k_6} c^{r_6} b^{t_6}$ *represent distinct elements in* $\mathcal{S}_m$.

Proof. 1. Assume that $c^j b^i = b^d c^r b^k$. Table 4.1 gives $j = r, k = d$ and $d = m$. Thus $cb = b^m cb$. Since cb is not contained in any of the relations in P, the relation $cb = b^m cb$ is not a consequence of the relations in P. Thus $c^j b^i \neq b^d c^r b^k$ in $\mathcal{S}_m$.

2. Applying $\alpha^q \gamma^j \beta^i$ and $\alpha^{q'} \beta^d \gamma^r \beta^k$ to p_8 implies that $q = q'$ and thus

$$a^q c^j b^i = a^q b^d c^r b^k.$$

It then follows that $a^2c^jb^i = a^2b^dc^rb^k$ which contradicts Rows 10 and 11 of Table 4.1.

3. Assume that $b^ic^ja^q = b^{i'}c^{j'}b^da^{q'}$. Then multiplying both sides by a from the left gives

$$a^{q+1}b^xc^y = a^{q'+1}b^{x'}c^{y'}b^z \quad \text{(Lemma (4.1.1))}$$

for some integers x, y, z, x', y'. But this implies that

$$11^{q+1}p_8 = (p_8)\alpha^{q+1}\beta^x\gamma^y = (p_8)\alpha^{q'+1}\beta^{x'}\gamma^{y'}\beta^z = 11^{q'+1}p_8 \Longrightarrow q = q'.$$

Thus, $a^2b^xc^y = a^2b^{x'}c^{y'}b^z$ which contradicts Rows 10 and 11 of Table 4.1. Thus

$$b^ic^ja^q \neq b^{i'}c^{j'}b^da^{q'}. \tag{4.3}$$

Similarly if $b^ic^ja^q = c^rb^ka^s$ then multiplying both sides by a from the left gives $a^{q+1}b^{i'}c^{j'} = a^{s+1}c^{r'}b^{k'}$ which contradicts Rows 10 and 12 of Table 4.1. Thus

$$b^ic^ja^q \neq c^rb^ka^s. \tag{4.4}$$

Also, if $b^ic^ja^q = b^{i'}c^{j'}b^da^{q'}$ then multiplying both sides by a from the left gives $a^{q+1}b^xc^y = a^{q'+1}b^{x'}c^{y'}b^{d'}$ which, again, contradicts Rows 10 and 11 Table 4.1. Thus

$$b^ic^ja^q \neq b^{i'}c^{j'}b^da^{q'}. \tag{4.5}$$

4. Assume that $b^ia^qb^dc^jb^k = b^{i'}a^{q'}c^{j'}b^{k'}$. Since Part 2 of this lemma implies that $a^qb^dc^jb^k \neq a^{q'}c^{j'}b^{k'}$ and there is no relation in P that allows b or c to jump over a to any side, it follows that $b^ia^qb^dc^jb^k = b^{i'}a^{q'}c^{j'}b^{k'}$ is not a consequence of the relations in P. Thus

$$b^ia^qb^dc^jb^k \neq b^{i'}a^{q'}c^{j'}b^{k'}. \tag{4.6}$$

To show 5, 6, 7 and 10 argue as in 4.

8. Assume that $c^j a^q b^i c^r b^d = c^{j'} a^{q'} c^{r'} b^{i'}$. Multiplying both sides by a from the left gives $a^{q+1} b^x c^y b^k = a^{q'+1} c^{y'} b^{k'}$ which contradicts Part 2 of this lemma. Thus

$$c^j a^q b^i c^r b^d \neq c^{j'} a^{q'} c^{r'} b^{i'} \tag{4.7}$$

9. Assume that $a^q b^i c^j = c^{j'} b^d a^{q'} b^k c^r$. Then multiplying both sides by b from the left gives $ba^q b^i c^j = bc^{j'} b^d a^{q'} b^k c^r$ which contradicts Rows 14 and 28 of Table 4.1.

Similarly if $a^q b^i c^j = b^{i'} c^{r'} a^s b^{d'} c^t$ then multiplying both sides by b from the left gives $ba^q b^i c^j = b^{i'+1} c^{r'} a^s b^{d'} c^t$ which contradicts rows 14 and 21 of Table 4.1.

To show that $c^{j'} b^d a^{q'} b^k c^r \neq b^{i'} c^{r'} a^s b^{d'} c^t$ argue as in 4. ∎

Lemma 4.1.4 *If a, b and c are as in P, then*

1. $b^i c^j b^d a^q b^k c^r = b^{i'} c^{j'} b^{d'} a^{q'} b^{k'} c^{r'} \iff i = i', j = j', d = d', q = q'$, $k = k', r = r'$,
2. $b^i c^j b^d a^q c^r b^k = b^{i'} c^{j'} b^{d'} a^{q'} c^{r'} b^{k'} \iff i = i', j = j', d = d', q = q'$, $r = r', k = k'$,
3. $b^i c^j b^d a^q b^k c^r b^t = b^{i'} c^{j'} b^{d'} a^{q'} b^{k'} c^{r'} b^{t'} \iff i = i',\ j = j',\ d = d'$, $q = q', k = k', r = r', t = t'$.

Proof. The action on p_8 of the corresponding words in $\{\alpha, \beta, \gamma\}^+$ of the above words gives $q = q'$ in each of the above cases. We will prove 3 and the proofs of the other cases follow by similar arguments. So, assume that

$$b^i c^j b^d a^q b^k c^r b^t = b^{i'} c^{j'} b^{d'} a^q b^{k'} c^{r'} b^{t'}. \tag{4.8}$$

Then Table 4.1 gives $i + d \equiv i' + d' \pmod m, j = j', k = k', r = r', t = t'$. Thus (4.8) becomes

$$b^i c^j b^d a^q b^k c^r b^t = b^{i'} c^j b^{d'} a^q b^k c^r b^t. \tag{4.9}$$

We will show that (4.9) implies that $b^ic^jb^d = b^{i'}c^jb^{d'}$. Assume that $b^ic^jb^d \neq b^{i'}c^jb^{d'}$. Since there is no relation in P that allows b or c to jump over a to either side, it follows that (4.9) is not a consequence of the relations in P and thus

$$b^ic^jb^da^qb^kc^rb^t \neq b^{i'}c^jb^{d'}a^qb^kc^rb^t$$

which contradicts (4.8). Thus, $b^ic^jb^d = b^{i'}c^jb^{d'}$ and therefore, Table 4.1 gives $i = i', d = d'$. ■

Theorem 4.1.2 *The set $N(\mathcal{S}_m)$, given in Theorem 4.1.1, is a normal form for $\mathcal{S}_m$.*

Proof. The first condition of Definition 3.1.1 follows by Theorem 4.1.1. Table 4.1, Proposition 3.2.1, Lemma 4.1.3 and Lemma 4.1.4 imply the second condition of Definition 3.1.1. Thus $N(\mathcal{S}_m)$ is a normal form for $\mathcal{S}_m$.

Theorem 4.1.3 *The order of the semigroup $\mathcal{S}_m$ is given by the following formula*

$$|\mathcal{S}_m| = m(1 + (m+1)^2) + 2[(m+1)(1 + (m+1)m)]^2.$$

Proof. Since $N(\mathcal{S}_m)$ is a normal form for $\mathcal{S}_m$, the order of $\mathcal{S}_m$ follows by counting the words in that normal form as they represent distinct elements in $\mathcal{S}_m$. ■

Remark 4.1.2 Note that $N(\mathcal{S}_m)$ can be written as follows:

$$N(\mathcal{S}_m) = \{b^ic^jb^k, b^ic^jb^ka^lb^dc^rb^t : 0 \leq i,j,k,d,r,t \leq m, 1 \leq l \leq 2\}$$

but we adopt the form given in Theorem 4.1.1 because it is easier to find the order of $\mathcal{S}_m$ from that form.

Next we study the idempotents of $\mathcal{S}_m$.

4.2 The set of idempotents of $\mathcal{S}_m$

In this section we will find the idempotents of the semigroup $\mathcal{S}_m$.

Remark 4.2.1 In what follows all the words are assumed to be in $N(\mathcal{S}_m)$ and, unless otherwise stated, for all $x, y \in \mathbf{N}$, $x \pmod y$ is in the range $1, \cdots, y$. Also, we recall that in Remark 4.1.1 we agreed that $x^{m-i} = x^m$ whenever $i = m$ and $x \in \{ b, c \}$.

Lemma 4.2.1 *Let b and c be as in the presentation of $\mathcal{S}_m$, and let $E(\mathcal{S}_m)$ be the set of idempotents in $\mathcal{S}_m$. Then, $b^i c^j b^k : 1 \leq i, j, k \leq m$ is in $E(\mathcal{S}_m)$ if and only if the following conditions hold:*

$$i + k \equiv m \pmod m \quad \textit{and} \quad j = m.$$

Proof. Assume that $b^i c^j b^k$ is in $E(\mathcal{S}_m)$. Then,

$$b^i c^j b^k = (b^i c^i b^k)^2 = b^{2i+k} c^{2j} b^k \quad (\text{ Lemma 4.1.1 and Corollary 4.1.1 })$$

$$\Longleftrightarrow 2i + k \equiv i \pmod m, \; 2j \equiv j \pmod m \quad (\text{ Theorem 4.1.2 })$$

$$\Longleftrightarrow i + k \equiv m \pmod m, \; j = m \quad (\, j \leq m) \quad (\text{see Remark 4.2.1}).$$

Conversely, if $i + k \equiv m \pmod m$ and $j = m$, then

$$(b^i c^m b^k)^2 = b^{2i+k} c^m b^k = b^i c^j b^k \quad (\text{Lemma 4.1.1 and Corollary 4.1.1}). \qquad \blacksquare$$

Remark 4.2.2 Note that allowing k to be zero in Lemma 4.2.1 implies that $b^i c^j$ is an idempotent if and only if $i = j = m$. On the other hand, allowing i to be zero implies that $c^j b^k$ is an idempotent if and only if $k = 0$ and $j = m$.

Lemma 4.2.2 *Let* $w = b^i c^j b^k a^q b^d c^r b^t : 0 \leq i, k, d, t, j, r \leq m, 1 \leq q \leq 2$ *be an idempotent. Then,*

1. $q = 2$;
2. $r = 0 \Longrightarrow j = 0$ *and* $i+k+d+t \equiv m \pmod m$ *(or* $i = k = d = t = 0$*)*;
3. $d = 0 \Longrightarrow i = k = t = 0$ *and* $j + r \equiv m \pmod m$ *(or* $j = r = 0$*)*.

Proof. 1. Either $q = 1$, or $q = 2$. Assume that $q = 1$. Then using Lemma 4.1.1 and the fact that w is an idempotent, it follows that $w^2 = xa^2y$ for some $x, y \in N(\mathcal{S}_m) \cap \{b, c\}^+$ and hence,

$$b^i c^j b^k a b^d c^r b^t = w = w^2 = xa^2 y.$$

It then follows from Theorem 4.1.2 that 1=2 which is a contradiction. Thus, $q = 2$.

2. Assume that $r = 0$. Then,

$$b^i c^j b^k a^2 b^{d+t} = w = w^2 = b^i c^j b^k a^2 b^{i+d+t} c^j b^{k+d+t} \quad \text{(Lemma 4.1.1)}$$

$$\Longrightarrow j = 0, \; i + k + d + t \equiv m \pmod m \quad \text{(Theorem 4.1.2)}.$$

3. If $j = 0 = r$ and $w = b^i a^2 b^d$, then $d = 0$ implies that

$$b^i a^2 = w = w^2 = b^i a^2 b^i a^2 = b^i a^2 b^i \text{ (Lemma 4.1.1)}$$

which implies that $i = 0$ (Theorem 4.1.2). On the other hand, if $\max(j, r) \geq 1$, then $d = 0$ and $r \neq 0$ (note that $j \neq 0$ implies $r \neq 0$ by 2) imply that

$$b^i c^j b^k a^2 c^r b^t = w = w^2 = b^i c^j b^k a^2 b^{i+k+t} c^{2r+j} b^t \quad \text{(Lemma 4.1.1)}.$$

It then follows from Theorem 4.1.2 and Remark 4.2.1 that

$$i + k + t = 0,\ j + 2r \equiv r \pmod m$$

$$\iff i = k = t = 0,\ j + r \equiv m \pmod{m}. \qquad \blacksquare$$

We now immediately deduce

Corollary 4.2.1 *Let w be as in Lemma 4.2.2, and let $d = 0$. Then, w is an idempotent if and only if the following conditions hold:*

1. $q = 2$;
2. $i = k = t = 0$;
3. *either* $j = r = 0$ *or* $j + r \equiv m \pmod{m}$.

Corollary 4.2.2 *Let w be as in Lemma 4.2.2, and let $r = 0$. Then, w is an idempotent if and only if the following conditions hold:*

1. $q = 2$;
2. $j = 0$;
3. *either* $i = k = d = t = 0$ *or* $i + k + d + t \equiv m \pmod{m}$.

Lemma 4.2.3 *The word*

$$w = b^i c^j b^k a^q b^d c^r b^t : 0 \le i, j, k, r \le m, 1 \le d, r \le m, 1 \le q \le 2;$$

is an idempotent if and only if the following conditions hold:

1. $q = 2$;
2. $i + k + d + t \equiv m \pmod{m}$;
3. $j + r \equiv m \pmod{m}$.

Proof. Assume that w is an idempotent. Then,

1. As in the proof of Lemma 4.2.2 (1).

2, 3. Since $d \neq 0 \neq r$, we get

$$b^i c^j b^k a^2 b^d c^r b^t = w = w^2 = b^i c^j b^k a^2 b^{(i+k+2d+t)} c^{(2r+j)} b^t \text{ (Lemma 4.1.1)}$$

which implies that

$$i + k + d + t \equiv m \pmod{m} \text{ and } r + j \equiv m \pmod{m}.$$

Conversely, assume that Conditions 1,2 and 3 hold. Then, it follows from Lemma 4.1.1 and Corollary 4.1.1 that

$$w^2 = b^i c^j b^k a^2 b^{i+k+2d+t} c^{j+2r} b^t = b^i c^j b^k a^2 b^d c^r b^t = w. \quad \blacksquare$$

Theorem 4.2.1 *The set of idempotents of $\mathcal{S}_m$ is as displayed in the following table:*

1	a^2	*11*	$b^i c^j a^2 b^{m-i} c^{m-j}$
2	$a^2 b^m$	*12*	$b^i c^j a^2 b^k c^{m-j} b^{m-(i+k)}$
3	$a^2 b^m c^m$	*13*	$b^i c^m b^{m-i}$
4	$a^2 b^i c^m b^{m-i}$	*14*	$b^i c^j b^k a^2 b^{(m-(i+k))} c^{m-j}$
5	$a^2 c^m$	*15*	$b^i c^j b^k a^2 b^d c^{m-j} b^{m-(i+k+d)}$
6	b^m	*16*	c^m
7	$b^i a^2 b^{m-i}$	*17*	$c^j a^2 b^m c^{m-j}$
8	$b^i a^2 b^{m-i} c^m$	*18*	$c^i a^2 b^j c^{m-i} b^{m-j}$
9	$b^i a^2 b^j c^m b^{m-(i+j)}$	*19*	$c^i a^2 c^{m-i}$, $c^j b^i a^2 b^{m-i} c^{m-j}$
10	$b^m c^m$	*20*	$c^j b^i a^2 b^k c^{m-j} b^{m-(i+k)}$

Table 4.2: $1 \leq i, j, k, d, r, t \leq m$.

Proof. The result follows by Lemma 4.2.1, Corollary 4.2.1, Corollary 4.2.2, Lemma 4.2.3, Remark 4.2.2 and the immediate fact that b^j is an idempotent if and only if $j = m$. $\blacksquare$

Corollary 4.2.3 *If I stands for the set of idempotents in $\mathcal{S}_m$, then by counting its normal form one finds that*

$$|I| = m^4 + 3m^3 + 4m^2 + 6m + 7.$$

4.3 The maximal subgroups of $\mathcal{S}_m$

In this section we will give a new method to find the maximal subgroups of a semigroup $\mathcal{S}$. Then we apply that method to find all the maximal subgroups of $\mathcal{S}_m$.

Definition 4.3.1 *Two words $u, v \in N(\mathcal{S}_m)$ are said to be of the same form if and only if there exist $i, j \in \mathbf{N}$ such that $u^i = v^j$.*

Definition 4.3.2 A relation $\mathcal{F}$ on $N(\mathcal{S}_m)$ is defined by the rule that $u \mathrel{\mathcal{F}} v$ if and only if u and v are of the same form.

The following lemma is a consequence of the previous two definitions.

Lemma 4.3.1 *The relation $\mathcal{F}$ is an equivalence relation.*

Notation 4.3.1 Let T be the subset of a semigroup $\mathcal{S}$ defined by:

$$T = \{\ x \in \mathcal{S} : (\exists\ r \in \mathbf{N}, r \geq 2,\ x^r = x)\ \}.$$

Also, if $x \in E(\mathcal{S})$, then G_x stands for the maximal subgroup of $\mathcal{S}$ containing the idempotent x.

The next theorem shows that any maximal subgroup of a semigroup $\mathcal{S}$ can be described in term of $\mathcal{F}$ and T.

Theorem 4.3.1 *If $x \in E(\mathcal{S})$, then*

$$G_x = [x]_{\mathcal{F}} \cap T$$

where $[x]_{\mathcal{F}}$ is the equivalence class of x with respect to $\mathcal{F}$.

Proof. Let $y \in G_x$. Since the identity element of G_x is x,

$$\exists\ r \in \mathbf{N} : y^r = e = x \Longrightarrow y \in [x]_{\mathcal{F}} \cap T \Longrightarrow G_x \subseteq [x]_{\mathcal{F}} \cap T. \qquad (4.10)$$

Conversely, if $y \in [x]_{\mathcal{F}} \cap T$ then

$$\exists r \in \mathbf{N} : y^r = x \ (x \text{ is an idempotent}). \qquad (4.11)$$

Since $y \in T$,

$$\exists\ t \in \mathbf{N} : (t \geq 2, y^t = y).$$

Thus, the kernel $K = \{y, y^2, \ldots, y^{t-1}\}$ is a subgroup contained in $[x]_{\mathcal{F}} \cap T$. Since $y \in K$, it follows that $y^r \in K \cap G_x$. Thus $K \subseteq G_x$ because G_x is a maximal subgroup ($\mathcal{H}$-class). Hence, $y \in G_x$ and $G_x = [x]_{\mathcal{F}} \cap T$. ∎

The following lemmas are necessary before we can describe the maximal subgroups of $\mathcal{S}_m$. Recall that $\mathcal{S}_m$ is defined by the presentation:

$$P = \langle\, a, b, c \mid a^3 = a, b^{m+1} = b, c^{m+1} = c, a^2ba = ab^{m-1}, a^2ca = ac^{m-1}, bc = cbc^m \,\rangle.$$

Lemma 4.3.2 *Let b and c be as in the presentation of $\mathcal{S}_m$, let $0 \leq i, i', k, k' \leq m$, and let $1 \leq j, j' \leq m$. Then, the words $w = b^i c^j b^k$ and $w' = b^{i'} c^{j'} b^{k'}$ are of the same form if and only if $k = k'$.*

Proof. Assume that w and w' are of the same form. Then

$$\begin{aligned}
&\exists\ r, r' \in \mathbf{N}\ : (b^i c^j b^k)^r = (b^{i'} c^{j'} b^{k'})^{r'} \\
&\Longrightarrow b^{ri+(r-1)k} c^{rj} b^k = b^{r'i'+(r'-1)k'} c^{r'j'} b^{k'} \ \ \text{(Lemma 4.1.1)} \\
&\Longrightarrow k = k' \ \ \text{(Theorem 4.1.2)}.
\end{aligned}$$

Conversely, if $k = k'$ then

$$w^m = (b^i c^j b^k)^m = b^{mi+(m-1)k} c^{mj} b^k = b^{m-k} c^m b^k = (b^{i'} c^{j'} b^k)^m = w'^m.$$

This shows that w and w' are of the same form. ∎

Lemma 4.3.3 *Let a, b and c be as in the presentation of $\mathcal{S}_m$. Then, the words $u = b^i c^j b^k a b^d c^r b^t$ and $v = b^i c^j b^k a^2 b^d c^r b^t : 0 \leq i, k, d, t, j \leq m$, $1 \leq r \leq m$ are of the same form.*

Proof. By Lemma 4.1.1 it follows that

$$\begin{aligned} u^2 &= b^i c^j b^k a^2 b^{m-i-k-t} c^{m-j} b^t; \\ v^m &= b^i c^j b^k a^2 b^{md+(m-1)(i+k+t)} c^{mr+(m-1)j} b^t \\ &= b^i c^j b^k a^2 b^{m-i-k-t} c^{m-j} b^t = u^2. \end{aligned}$$

Hence, u and v are of the same form. ∎

Lemma 4.3.4 *The words $u = b^i c^j b^k a b^d$ and $v = b^i c^j b^k a^2 b^d : 0 \leq i, k, d \leq m$, $1 \leq j \leq m$ are not groups elements.*

Proof. Assume that u and v are groups elements. Then, by Theorem 4.3.1 $u, v \in T$, that is, there exist $r, s \in \mathbf{N} : (r, s \geq 2, u^r = u, v^s = v)$. But Lemma 4.1.1 implies that

$$u^r = \begin{cases} b^i c^j b^k a^2 b^{m-i-d} c^{m-j} b^{m-k+d} : \max(k, d) \geq 1, \max(i, d) \geq 1, r \text{ even}, \\ c^j b^k a^2 c^{m-j} b^{m-k} : k \neq 0 = i = d, r \text{ even}, \\ c^j a^2 c^{m-j} : i = d = k = 0, r \text{ even}, \\ b^i c^j b^k a b^k c^m b^{m-k+d} : \max(k, d) \geq 1, r \text{ odd}, \\ b^i c^j a c^{m-j} : k = d = 0 \neq i, r \text{ odd}, \end{cases}$$

which shows that for each $r \geq 2$, $u^r = w a^q b^t c^j b^s : w \in \{b, c\}^+ \cap N(\mathcal{S}_m)$, $1 \leq j \leq m$, $1 \leq q \leq 2, 0 \leq t, s \leq m$. Since $1 \leq j$, it follows from Theorem

4.1.2 that $u^r \neq u$ for any $r \geq 2$ which is a contradiction. Similarly, Lemma 4.1.1 implies that

$$v^s = \begin{cases} b^i c^j b^k a^2 b^{(s-1)(i+d)+(s-2)k} c^{(s-1)j} b^{k+d} : s \text{ even}, \\ b^i c^j b^k a b^{(s-1)(i+d)+(s-2)k} c^{(s-1)j} b^{k+d} : s \text{ odd}, \end{cases}$$

since $s \geq 2$, it follows that $v^s = wa^q b^t c^j b^s : w \in \{b, c\}^+ \cap N(\mathcal{S}_m)$, $1 \leq j \leq m$, $1 \leq q \leq 2, 0 \leq t, s \leq m$, and thus $v^s \neq v$ for any $s \geq 2$. This conclude that u and v are not groups elements. ∎

Remark 4.3.1 1. Since the words u and v of Lemma 4.3.4 are not groups elements, they are not considered in the rest of this chapter.
2. It is a routine job to show that $b^i ab^k, b^i a^2 b^k : 0 \leq i, k \leq m$ are of the same form.

Lemma 4.3.5 *Let $u_1, u_2, v_1, v_2 \in \{ b, c \}^+ \cap N(\mathcal{S}_m)$. Then the words $u = u_1 a^q u_2$ and $v = v_1 a^{q'} v_2$ are of the same form if and only if the following conditions hold :*

1. *$u_1 = v_1$,*
2. *u_1u_2 and u_1v_2 are of the same form,*
3. *u_2u_1 and v_2u_1 are of the same form.*

Proof. By Lemmas 4.3.3, 4.3.4 and Remark 4.3.1 it is enough to show this lemma for $u = u_1a^2u_2$ and $v = v_1a^2v_2$. So assume that $u = u_1a^2u_2$, $v = v_1a^2v_2$ are of the same form. Then,

$$\exists\ r, s \in \mathbf{N} : u^r = v^s.$$

It then follows by Lemma 4.1.1 that

$$u_1a^2u_2(u_1u_2)^r = v_1a^2v_2(v_1v_2)^s.$$

It then follows from Theorem 4.1.2 that

$$u_1 = v_1,\ u_2(u_1u_2)^r = v_2(u_1v_2)^s \tag{4.12}$$

which implies that

$$(u_1u_2)^{r+1} = (u_1v_2)^{s+1}.$$

Thus, Conditions 1 and 2 hold. Also, (4.12) implies that

$$(u_2u_1)^r u_2 = (v_2u_1)^s v_2 \Longrightarrow (u_2u_1)^{r+1} = (v_2u_1)^{s+1}.$$

Thus, Conditions 1, 2 and 3 hold.

Conversely, if Conditions 1, 2 and 3 hold then

$$\exists\ r, r', s, s' \in \mathbf{N} : (u_1u_2)^r = (u_1v_2)^s,\ (u_2u_1)^{r'} = (v_2u_1)^{s'}.$$

Let $h = r + s - 1$. Then,

$$\begin{aligned} u^{r+s-1} &= u_1a^2u_2(u_1u_2)^{s+r-1} = u_1a^2u_2(u_1u_2)^s(u_1u_2)^{r-1} \\ &= u_1a^2(u_2u_1)^s u_2(u_1u_2)^{r-1} = v_1a^2(v_2v_1)^{s'}u_2(v_1u_2)^{r-1} \\ &= v_1a^2v_2(v_1v_2)^{s'-1}v_1(u_2(v_1u_2)^{r-1}) = v_1a^2v_2(v_1v_2)^{s'-1}(v_1v_2)^{r'} \\ &= v_1a^2v_2(v_1v_2)^{s'+r'-1} = v^{r'+s'-1}. \end{aligned}$$

Hence u and v are of the same form.

Example 4.3.1 The words bc and bcb are not of the same form.

Proof. Follows by Lemma 4.3.2. To verify, assume that they are of the same form. Then,

$$\begin{aligned} &\exists\ q, h \in \mathbf{N} : (bc)^q = (bcb)^h \Longrightarrow b^qc^q = b^{2h-1}c^hb \ \ \text{(Lemma 4.1.1)} \\ &\Longrightarrow b^qc^qb = b^{2h-1}c^hb^2 \Longrightarrow 1 = 2 \ \ \text{(Theorem 4.1.2)} \end{aligned}$$

which is a contradiction. Thus cb and bcb are not of the same form. ∎

Example 4.3.2 The words $w_1 = cab$ and $w_2 = cabcb$ are of the same form but w_1 is not a group element.

Proof. By Lemmas 4.3.5 and 4.3.2 it follows that w_1 and w_2 are of the same form. Since $w_1^{2i} = ca^2b^{m-1}c^{m-1}b,\ w_1^{2i+1} = cab^mc^mb : i \in \mathbf{N},\ w_1^t \neq w_1$ for any $t \in \mathbf{N}$. Hence, $w_1 \notin T$. By Theorem 4.3.1 w_1 is not a group element. ■

The next example is an immediate consequence of Lemma 4.3.2.

Example 4.3.3 The words bcb and bcb^2 are not of the same form.

Example 4.3.4 The maximal subgroup containing $b^{i_0}c^mb^{m-i_0} : 1 \leq i_0 \leq m$ is

$$G_{b^{i_0}c^mb^{m-i_0}} = \{\ b^ic^jb^{m-i_0} : 1 \leq i, j \leq m\ \}.$$

Proof. By Lemma 4.3.2

$$[b^{i_0}c^mb^{m-i_0}]_{\mathcal{F}} = \{\ b^ic^jb^{m-i_0} : 0 \leq i \leq m,\ 1 \leq j \leq m\ \}.$$

By Theorem 4.3.1 $G_{b^{i_0}c^mb^{m-i_0}} = [b^{i_0}c^mb^{m-i_0}]_{\mathcal{F}} \cap T$. Hence, if w is in $G_{b^{i_0}c^mb^{m-i_0}}$ then $w = b^ic^jb^{m-i_0} : 0 \leq i \leq m,\ 1 \leq j \leq m$ and

$$\exists\ r \in \mathbf{N}(r \geq 2) : w^r = w \qquad\qquad (*).$$

It then follows from Lemma 4.1.1 that

$$w^r = b^{ri+(r-1)(m-i_0)}c^{rj}b^{m-i_0} = b^ic^jb^{m-i_0}$$

$$\Longrightarrow i \geq 1 \Longrightarrow w = b^ic^jb^{m-i_0} : 1 \leq i, j \leq m$$

$$\Longrightarrow G_{b^ic^jb^{m-i_0}} \subseteq \{b^ic^jb^{m-i_0} : 1 \leq i, j \leq m\}.$$

But $\{b^ic^jb^{m-i_0} : 1 \leq i, j \leq m\}$ is a group and $G_{b^ic^jb^{m-i_0}}$ is a maximal. It then follows that $G_{b^ic^jb^{m-i_0}} = \{b^ic^jb^{m-i_0} : 1 \leq i, j \leq m\}$. ■

Example 4.3.5 The maximal subgroup containing the idempotent e where

$$e = b^{i_0}c^{j_0}b^{k_0}a^2b^{d_0}c^{m-j_0}b^{m-(i_0+k_0+d_0)} : 1 \leq i_0, j_0, k_0, d_0 \leq m$$

is

$$G_e = \{\ b^{i_0}c^{j_0}b^{k_0}a^q b^i c^r b^{m-(i_0+k_0+d_0)} \ : 1 \leq i, r \leq m, 1 \leq q \leq 2\}.$$

Proof. Lemmas 4.3.3, 4.3.4, 4.3.5 and 4.3.2 imply that

$$[e]_{\mathcal{F}} = \{\ b^{i_0}c^{j_0}b^{k_0}a^q b^d c^r b^{m-(i_0+k_0+d_0)} : 0 \leq d, r \leq m; 1 \leq q \leq 2\ \}.$$

Also, Theorem 4.3.1 implies that if $w = b^{i_0}c^{j_0}b^{k_0}a^q b^d c^r b^{m-(i_0+k_0+d_0)}$ is in G_e then there exists an integer $r_0 \geq 2$ such that $w^{r_0} = w$, that is

$$w^{r_0} = b^{i_0}c^{j_0}b^{k_0}a^2b^{r_0d+(r_0-1)d_0}c^{r_0r+(r_0-1)j_0}b^{m-(i_0+k_0+d_0)} \tag{4.13}$$

$$= b^{i_0}c^{j_0}b^{k_0}a^2b^d c^r b^{m-(i_0+k_0+j_0)} \text{ (Lemma 4.1.1)}$$

$$\Longrightarrow 1 \leq d, r \leq m \text{ (Theorem 4.1.2)}.$$

Also, it follows from (4.13) that $w^{m+1} = w$ and the result follows. ∎

Theorem 4.3.2 *The possible maximal subgroups contained in* $N(\mathcal{S}_m) \cap \{b, c\}^+$ *(up to isomorphism) are*

1. $\langle\ x\ :\ x^m = 1\rangle$,
2. $\langle\ x, y\ :\ x^m = y^m = xyx^{-1}y^{-1} = 1\ \rangle$.

Proof. By Theorem 4.2.1, Lemma 4.3.2 and Theorem 4.3.1 the maximal subgroups contained in $\mathcal{S}_m \cap \{b, c\}^+$ are

i. $\{b^i \;:\; 1 \leq i \leq m\}$,
ii. $\{c^i \;:\; 1 \leq i \leq m\}$,
iii. $\{b^i c^j \;:\; 1 \leq i, j \leq m\}$,
iv. $\{b^i c^j b^{m-i_0} \;:\; 1 \leq i, j \leq m\} \;:\; 1 \leq i_0 \leq m-1\}$.

Clearly i and ii are of type 1.

iii. Let G be the group defined by the presentation in (2), and let G_0 be the group in *iii*. Let $x_0 = b^m c$, $y_0 = bc^m$. Since

$$b^i c^j = (b^m c)^j (bc^m)^i = x_0^j y_0^i \text{ (Lemma 4.1.1)},$$

it follows that G_0 is generated by x_0 and y_0. Also x_0 and y_0 are in G_0 and thus $\langle\, x_0,\; y_0 \,\rangle \subseteq G_0$. Thus $\langle\, x_0,\; y_0 \,\rangle = G_0$.

Since x_0 and y_0 satisfy the relations in the presentation of G, the mapping $\phi : G \longrightarrow \langle\, x_0,\; y_0 \,\rangle$ defined by

$$(x^{i_1} y^{i_2} \cdots x^{i_{q-1}} y^{i_q})\phi = x_0^{i_1} y_0^{i_2} \cdots x_0^{i_{q-1}} y_0^{i_q} : 0 \leq i_j \leq m$$

is an onto homomorphism (note that ϕ is the homomorphic extension of $f : \{x, y\} \longrightarrow \langle x_0, y_0 \rangle$ which takes $x \mapsto x_0$ and $y \mapsto y_0$). Moreover, ϕ is injective because if g_1 and g_2 are in G with $(g_1)\phi = (g_2)\phi$ then

$$g_1 = x^i y^j,\; g_2 = x^d y^r \Longrightarrow x_0^i y_0^j = x_0^d y_0^r \Longrightarrow b^i c^j = b^d c^r \text{ (Lemma 4.1.1)}$$

$$\Longrightarrow i = d,\; j = d \Longrightarrow g_1 = g_2 \text{ (Theorem 4.1.2)} \Longrightarrow G \cong \langle\, x_0,\; y_0 \,\rangle.$$

Thus iii is of type 2.

Similarly, iv is of type 2 with $x = b^{i_0+1} c b^{m-i_0}$, $y = b^{i_0+1} c^m b^{m-i_0}$. ∎

The next theorem gives the maximal subgroups contained in the complement of $\mathcal{S}_m \cap \{b, c\}^+$ up to isomorphism.

Theorem 4.3.3 *The possible maximal subgroups contained in the complement of* $N(\mathcal{S}_m) \cap \{b, c\}^+$ *are (up to isomorphism)*

1. $\langle\ a:\ a^2=1\},$

2. $\langle x,y : x^2=y^m=(xy)^2=1\ \rangle,$

3. $\langle x,y,z\ : x^m=y^m=z^2=(xy)^m=(xz)^2=(yz)^2=(xyz)^2=1\rangle.$

Proof. The maximal subgroups of $(N(\mathcal{S}_m)\cap\{b,c\}^+)^c$ are

$$G_x : x\in E(\mathcal{S}_m)\setminus\{b,c\}^+.$$

One can find $G_x : x\in E(N(\mathcal{S}_m)\setminus\{b,c\}^+$ as we did in Example 4.3.4 and Example 4.3.5. Clearly, $G_{a^2}=\{\ a,a^2\ \}$ is of type 1. To show that $G_{a^2b^m}$ is of type 2, note that

$$G_{a^2b^m}=\{\ a^ib^j : 1\le i,j\le m\ \}.$$

By taking $x_0=ab^m,\ y_0=a^2b$, we get

$$a^ib^j=(ab^m)^i(a^2b)^j.$$

Thus, x_0 and y_0 generate $G_{a^2b^m}$. Therefore, $G_{a^2b^m}\subseteq\langle\ x_0,\ y_0\ \rangle$. Since x_0 and y_0 are in $G_{a^2b^m}$, $\langle\ x_0,\ y_0\ \rangle\subseteq G_{a^2b^m}$ and thus $G_{a^2b^m}=\langle\ x_0,\ y_0\ \rangle$. Let G be the group defined by the presentation in (2). The generators of $G_{a^2b^m}$, x_0 and y_0, satisfy the relations in (2) because

$$x_0^2=abab=a^2b^{m-1}b=a^2b^m\quad\text{(Lemma 4.1.1)},$$

$$y_0^m=(a^2b^{m-1})^m=a^2b^m\quad\text{(Lemma 4.1.1)},$$

and

$$(x_0y_0)^2=(aba^2b^{m-1})^2=(ab^m)^2=a^2b^m.$$

Therefore the mapping $\phi : G\longrightarrow G_{a^2b^m}$ defined by

$$(x^{i_1}y^{i_2}\cdots x^{i_{q-1}}y^{i_q})\phi=x_0^{i_1}y_0^{i_2}\cdots x_0^{i_{q-1}}y_0^{i_q} : 0\le i_j\le m$$

is an onto homomorphism (note that ϕ is the homomorphic extension of $f : \{x, y\} \longrightarrow \langle x_0, y_0\rangle$ which takes $x \mapsto x_0$ and $y \mapsto y_0$).

If $g \in G$, then $g = x^i y^j : \ 1 \le i \le 2, 1 \le j \le m$ because

$$(xy)^2 = 1, x^2 = 1 \Longrightarrow xyx = y^{m-1} \Longrightarrow xy^j x = y^{m-j} \Longrightarrow xy^j = y^{m-j}x.$$

Similarly, $yxy = x$ and thus $y^j xy^r = y^{\max(0,j-r)}xy^{\max(0,r-j)}$. It then follows that

$$\forall\, g \in G,\ g = x^i y^j : 1 \le i \le 2, 1 \le j \le m.$$

Hence, if $g_1, g_2 \in G : (g_1)\phi = (g_2)\phi$ then

$$g_1 = x^i y^j, g_2 = x^d y^k \Longrightarrow x_0^i y_0^j = x_0^d y_0^r \Longrightarrow a^i b^j = a^d b^k.$$

It then follows from Theorem 4.1.2 that $i = d, j = k$ and thus $g_1 = g_2$. Hence, ϕ is an isomorphism and

$$G_{a^2b^m} \cong G(\langle\ x,\ y : x^2 = y^m = (xy)^2 = 1\ \rangle).$$

Similarly $G_{a^2c^m}$ is also of type 2 with $x = ac$ and $y = a^2c^{m-1}$.

For $G_{b^{i_0}a^2b^{m-i_0}}$ take $x_0 = b^{i_0}ab^m,\ y_0 = b^{i_0}a^2b^{m+1-i_0}$. First, note that if q is the smallest integer for which $y_0^q = b^{i_0}a^2b^{m-i_0}$ then $q = m$ because

$$b^{i_0}a^2b^{m-i_0} = y_0^q = b^{i_0}a^2b^{q-i_0} \text{ (Lemma 4.1.1)}$$

and Theorem 4.1.2 implies that $q - i_0 \equiv m - i_0 \pmod m$ and since $q \le m$, we must have $q = m$. Also,

$$b^{i_0}a^q b^i = (b^{i_0}ab^m)^q (b^{i_0}a^2b^{m+1-i_0})^i = x_0^q y_0^i$$

i.e. x_0 and y_0 generate $G_{b^{i_0}a^2b^{m-i_0}}$. Furthermore, x_0 and y_0 satisfy the relations in (2) because

$$x_0^2 = b^{i_0}ab^m b^{i_0}ab^m = b^{i_0}a^2b^{m-i_0} \text{ (Lemma 4.1.1)}$$

and

$$y_0^m = b^{i_0}a^2b^{m-i_0} = (x_0y_0)^2.$$

Hence, as we did for $G_{a^2b^m}$, one can easily show that

$$G_{b^{i_0}a^2b^{m-i_0}} \cong G(\langle\ x, y : x^2 = y^m = (xy)^2 = 1\rangle).$$

Now consider $G_{a^2b^mc^m}$. Let P be the presentation in (3). As in Examples 4.3.4 and 4.3.5, one can show that

$$G_{a^2b^mc^m} = \{\ a^ib^jc^k : 1 \leq i \leq 2, 1 \leq j, k \leq m\ \}.$$

If we take $x_0 = a^2b^mc,\ y_0 = a^2bc^m$ and $z_0 = ab^mc^m$, then

$$a^2b^ic^j = (x_0)^j(y_0)^i \text{ (Lemma 4.1.1)},$$

$$ab^ic^j = z_0(x_0)^j(y_0)^i \text{ Lemma 4.1.1)}.$$

Thus x_0, y_0 and z_0 generate $G_{a^2b^mc^m}$ and since they are in $G_{a^2b^mc^m}$,

$$G_{a^2b^mc^m} = \ \langle\ x_0,\ y_0,\ z_0\rangle.$$

By Lemma 4.1.1, it follows that x_0, y_0 and z_0 satisfy the relations in P. Thus the mapping $\phi : G(P) \longrightarrow G_{a^2b^mc^m}$ defined by

$$(x^{i_1}y^{i_2}z^{i_3}\cdots x^{i_{q-2}}y^{i_{q-1}}z^{i_q})\phi = x_0^{i_1}y_0^{i_2}z_0^{i_3}\cdots x_0^{i_{q-2}}y_0^{i_{q-1}}z_0^{i_q} : 0 \leq i_j \leq m$$

is an onto homomorphism (note that ϕ is the homomorphic extension of $f : \{x, y\} \longrightarrow \langle x_0, y_0\rangle$ which takes $x \mapsto x_0$ and $y \mapsto y_0$). Also, any word w in $G(P)$ is of the form $z^qx^iy^j$ because

$$xzx = z, yzy = z, zyz = y^{m-1}, zxz = x^{m-1}$$

$$\Longrightarrow zxyz = zxz^2yz = zxzy^{m-1} = x^{m-1}y^{m-1}, xy = yx.$$

Thus, if $g_1, g_2 \in G(P) : (g_1)\phi = (g_2)\phi$ then

$$g_1 = z^dx^iy^j, g_2 = z^rx^ky^t \Longrightarrow z_0^dx_0^iy_0^j = z_0^rx_0^ky_0^t$$

$$\Longrightarrow a^d b^j c^i = a^r b^t c^k \Longrightarrow d = r, j = t, i = k \Longrightarrow g_1 = g_2.$$

Hence $G_{a^2 b^m c^m} \cong G(P)$.

Similar arguments show that the other groups are of type 3. They are listed in the following table along with their generators:

No	Group	Generators
1	$G_{a^2 b^{i_0} c^m b^{m-i_0}}$	$x = a^2 b^m c b^{m-i_0}, y = a^2 b c^m b^{m-i_0}, z = a b^m c^m b^{m-i_0}$
2	$G_{b^i a^2 b^{m-i} c^m}$	$x = b^i a^2 b^m c,\ y = b^i a^2 b c^m,\ z = b^i a b^m c^m$
3	$G_{c^i a^2 b^m c^{m-i}}$	$x = c^i a^2 b^m c,\ y = c^i a^2 b c^m,\ z = c^i a b^m c^m$
4	$G_{b^i a^2 b^j c^m b^{m-(i+j)}}$	$x = b^i a^2 b^m c b^{m-i-j},\ y = b^i a^2 b c^m b^{m-i-j}$, $z = b^i a b^m c^m b^{m-i-j}$
5	$G_{c^j b^i a^2 b^k c^{m-j} b^{m-(i+k)}}$	$x = c^j b^i a^2 b^m c b^{m-i-k},\ y = c^j b^i a^2 b c^m b^{m-i-k}$, $z = c^j b^i a b^m c^m b^{m-i-k}$
6	$G_{b^i c^j b^k a^2 b^{m-(i+k)} c^{m-j}}$	$x = b^i c^j b^k a^2 b^m c,\ y = b^i c^j b^k a^2 b^m c, z = b^i c^j b^k a b^m c^m$
7	$G_{b^i c^j a^2 b^{m-i} c^{m-j}}$	$x = b^i c^j a^2 b^m c, y = b^i c^j a^2 b c^m, z = b^i c^j a b^m c^m$
8	$G_{b^i c^j b^k a^2 b^d c^{m-j} b^{m-(i+k+d)}}$	$x = b^i c^j b^k a^2 b^m c b^{m-i-k-d},\ y = b^i c^j b^k a^2 b c^m b^{m-i-k-d}$, $z = b^i c^j b^k a b^m c^m b^{m-i-k-d}$
9	$G_{c^i a^2 b^j c^{m-i} b^{m-j}}$	$x = c^i a^2 b^m c b^{m-j}, y = c^i a^2 b c^m b^{m-j}, z = c^i a b^m c^m b^{m-j}$
10	$G_{b^i c^j a^2 b^k c^{m-j} b^{m-(i+k)} \pmod m}$	$x = b^i c^j a^2 b^m c b^{m-i-k},\ y = b^i c^j a^2 b c^m b^{m-i-k}$, $z = b^i c^j a b^m c^m b^{m-i-k}$

Table 4.3: The maximal subgroups of $\mathcal{S}_m$.

4.4 The Semigroups $\mathcal{S}_{(\ell,m,n)}$

In this section, we generalize the results that we have obtained in the previous sections to a generalized form of $\mathcal{S}_m$ which is presented by:

$$P = \langle\ a, b, c : a^{\ell+1} = a, b^{m+1} = b, c^{n+1} = c, a^{\ell} ba = ab^{m-1},$$

$$a^{\ell} ca = ac^{n-1}, bc = cbc^n\ \rangle$$

where ℓ, m and n are greater than or equal to 2.

The semigroup $\mathcal{S}_{(\ell,m,n)}$ has several types of structure depending on whether ℓ is even or odd. We start by investigating the case in which ℓ is even.

4.4.1 When ℓ is even.

In this section ℓ stands for an even integer greater than 1.

The next lemma gives some useful relations to find a normal form for $\mathcal{S}_{(\ell,m,n)}$.

Lemma 4.4.1 *Let* a, b *and* c *be as in* P. *Then,*

1. $ab^ia^\ell = ab^i$, $ac^ia^\ell = ac^i$,
2. $ab^ia = a^2b^{m-i}$, $ac^ia = a^2c^{n-i}$,
3. $ab^ia^{2t} = a^{2t+1}b^i$, $ac^ia^{2t} = a^{2t+1}c^i$,
4. $ab^ia^{2t+1} = a^{2t+2}b^{m-i}$, $ac^ia^{2t+1} = a^{2t+2}c^{n-i}$,
5. $ab^ic^jb^da^{2t} = a^{2t+1}b^ic^jb^d$,
6. $ab^ic^jb^da^{2t+1} = a^{2t+2}b^{m-i}c^{n-j}b^{m-d}$,
7. $c^ib^jc^k = b^jc^{i+k}$.

Proof. The proof is as in the proof of Lemma 4.1.1 . ∎

Remark 4.4.1 The relations in the previous lemma don't depend on whether ℓ is even or odd. Hence, the lemma is true for both cases.

The next theorem is a consequence of the previous lemma.

Theorem 4.4.1 *Let $N(\mathcal{S}_{(\ell,m,n)})$ be the subset of $\{ a,b,c \}^+$ defined by*

$$N(\mathcal{S}_{(\ell,m,n)}) = \{ b^i c^j b^k, b^i c^j b^k a^q b^d c^r b^t : 0 \leq i,k,d,t \leq m, 0 \leq j,r \leq n, 1 \leq q \leq \ell \}.$$

Then, for any $w \in \{ a,b,c\}^+$ there exists $w^* \in N(\mathcal{S}_{(\ell,m,n)})$ such that $w = w^*$ in $\mathcal{S}_{(\ell,m,n)}$.

Proof. The result follows by Lemma 4.4.1. ■

Corollary 4.4.1 *The semigroup $\mathcal{S}_{(\ell,m,n)}$ is finite.*

Slight modifications of the mappings α, β and γ, given in Section 1 of this chapter, with some modifications of Lemma 4.1.3 and Lemma 4.1.4 imply that the words in $N(\mathcal{S}_{(\ell,m,n)})$ represent distinct elements in $\mathcal{S}_{(\ell,m,n)}$. Hence, we have the following theorem:

Theorem 4.4.2 *The set $N(\mathcal{S}_{(\ell,m,n)})$ is a normal form for $\mathcal{S}_{(\ell,m,n)}$.*

Theorem 4.4.3 *If ℓ is even, then*

$$|\mathcal{S}_{(\ell,m,n)}| = m + (m+1)^2 n + \ell(m+1)^2(1+(m+1)n)^2.$$

Proof. The results follow by counting the elements in $N(\mathcal{S}_{(\ell,m,n)})$. ■

The Idempotents of $\mathcal{S}_{(\ell,m,n)}$

Lemma 4.2.1, Remark 4.2.2, Lemma 4.2.2, Corollaries 4.2.1, 4.2.2 and Lemma 4.2.3 hold in $\mathcal{S}_{(\ell,m,n)}$ with slight modifications. Hence, we have the following theorem:

Theorem 4.4.4 *The set of idempotents of $\mathcal{S}_{(\ell,m,n)}$ is as displayed in the following table:*

1	a^ℓ	11	$b^i c^j a^\ell b^{m-i} c^{n-j}$
2	$a^\ell b^m$	12	$b^i c^j a^\ell b^k c^{n-j} b^{m-(i+k)}$
3	$a^\ell b^m c^n$	13	$b^i c^n b^{m-i}$
4	$a^\ell b^i c^n b^{m-i}$	14	$b^i c^j b^k a^\ell b^{(m-(i+k))} c^{n-j}$
5	$a^\ell c^n$	15	$b^i c^j b^k a^\ell b^d c^{n-j} b^{m-(i+k+d)}$
6	b^m	16	c^n
7	$b^i a^\ell b^{m-i}$	17	$c^j a^\ell b^m c^{n-j}$
8	$b^i a^\ell b^{m-i} c^n$	18	$c^i a^\ell b^j c^{n-i} b^{m-j}$
9	$b^i a^\ell b^j c^n b^{m-(i+j)}$	19	$c^i a^\ell c^{n-i},\ c^j b^i a^\ell b^{m-i} c^{m-j}$
10	$b^m c^n$	20	$c^j b^i a^\ell b^k c^{n-j} b^{m-(i+k)}$

Table 4.4 $:1 \leq i, k, d, t \leq m,\ 1 \leq j, r \leq n.$

The maximal subgroups contained in $\mathcal{S}_{(\ell,m,n)}$.

Since Lemmas 4.3.2, 4.3.4, 4.3.5 and Remark 4.3.1 hold in $S_{(\ell,m,n)}$, the maximal subgroups of $\mathcal{S}_{(\ell,m,n)}$ are as in the following theorem:

Theorem 4.4.5 *The maximal subgroups of $\mathcal{S}_{(\ell,m,n)}$ are (up to isomorphism):*

1. $\langle\ x : x^\ell = 1\ \rangle,\ \langle\ x\ : x^m = 1\ \rangle,\ \langle\ x : x^n = 1\ \rangle.$
2. $\langle\ x,\ y\ : x^m = y^n = xyx^{-1}y^{-1} = 1\ \rangle,$
3. $\langle\ x,\ y\ : x^\ell = y^m = (xy)^\ell = 1\ \rangle,$
4. $\langle\ x,\ y\ : x^\ell = y^n = (xy)^\ell = 1\ \rangle,$
5. $\langle\ x,\ y,\ z\ : x^\ell = y^m = z^n = (xy)^\ell = (xz)^\ell = (xyz)^\ell = yzy^{-1}z^{-1} = yzxyzx^{-1} = 1\ \rangle.$

Proof. The proof is similar to the proof of Theorem 4.3.2 and Theorem 4.3.3. ■

We summarize the detail of the maximal subgroups of $\mathcal{S}_{(\ell,m,n)}$ in the following table:

No	Subgroup	P	Generators
1	G_{a^ℓ}	1	$x = a$
2	$G_{a^\ell b^m}$	3	$x = ab^m,\ y = a^\ell b$
3	$G_{a^\ell b^m c^n}$	5	$x = ab^m c^n,\ y = a^\ell bc^n,\ z = a^\ell b^m c$
4	$G_{a^\ell b^i c^n b^{m-i}}$	5	$x = ab^m c^n b^{m-i},\ y = a^\ell bc^n b^{m-i},\ z = a^\ell b^m cb^{m-i}$
5	$G_{a^\ell c^n}$	4	$x = ac^n, y = a^\ell c$
6	G_{b^m}	1	$x = b$
7	$G_{b^i a^\ell b^{m-i}}$	1	$x = b^i ab^m, y = b^i a^\ell b$
8	$G_{b^i a^\ell b^{m-i} c^n}$	5	$x = b^i ab^m c^n,\ y = b^i a^\ell bc^n,\ z = b^i a^\ell b^m c$
9	$G_{b^i a^\ell b^j c^n b^{m-(i+j)}}$	5	$x = b^i ab^m c^n b^{m-(i+j)},\ y = b^i a^\ell bc^n b^{m-(i+j)},$ $z = b^i a^\ell b^m cb^{m-(i+j)}$
10	$G_{b^m c^n}$	2	$x = bc^n,\ y = b^m c$
11	$G_{b^i c^j a^\ell b^{m-i} c^{n-j}}$	5	$x = b^i c^j ab^m c^n,\ y = b^i c^j a^\ell bc^n,\ z = b^i c^j a^\ell b^m c$
12	$G_{b^i c^j a^\ell b^k c^{n-j} b^{m-(i+k)}}$	5	$x = b^i c^j ab^m c^n b^{m-(i+k)},\ y = b^i c^j a^\ell bc^n b^{m-(i+k)},$ $z = b^i c^j a^\ell b^m cb^{m-(i+k)}$
13	$G_{b^i c^n b^{m-i}}$	2	$x = bc^n b^{m-i},\ y = b^m cb^{m-i}$
14	$G_{b^i c^j b^k a^\ell b^{(m-(i+k))} c^{n-j}}$	5	$x = b^i c^j b^k ab^m c^n,\ y = b^i c^j b^k a^\ell bc^n,\ z = b^i c^j b^k a^\ell b^m c$
15	$G_{b^i c^j b^k a^\ell b^d c^{n-j} b^{m-(i+k+d)}}$	5	$x = b^i c^j b^k ab^m c^n b^{m-(i+k+d)},\ y = b^i c^j b^k a^\ell bc^n b^{m-(i+k+d)},$ $z = b^i c^j b^k a^\ell b^m cb^{m-(i+k+d)}$
16	G_{c^n}	1	$x = c$
17	$G_{c^j a^\ell b^m c^{n-j}}$	5	$x = c^j ab^m c^n, y = c^j a^\ell bc^n, z = c^j a^\ell b^m c$
18	$G_{c^j a^\ell b^i c^{n-j} b^{m-i}}$	5	$x = c^j ab^m c^n b^{m-i}, y = c^j a^\ell bc^n b^{m-i}, z = c^j a^\ell b^m cb^{m-i}$
19	$G_{c^i a^\ell c^{n-i}}$	4	$x = c^j ac^n, y = c^j a^\ell c$
20	$G_{c^j b^i a^\ell b^k c^{n-j} b^{m-(i+k)}}$	5	$x = c^j b^i ab^m c^n b^{m-(i+k)}, y = c^j b^i a^\ell bc^n b^{m-(i+k)},$ $z = c^j b^i a^\ell b^m cb^{m-(i+k)}$

Table 4.5: $1 \leq i, k, d, t \leq m,\ 1 \leq j, r \leq n$.
The maximal subgroups of $\mathcal{S}_{(\ell,m,n)} : \ell$ is even.

Next we consider the case in which ℓ is odd.

4.4.2 The semigroup $\mathcal{S}_{(\ell,m,n)} : \ell$ is odd.

In what follows ℓ stands for an odd integer greater than 1.

The next lemma gives some other properties of $\mathcal{S}_{(\ell,m,n)}$ when ℓ is an odd integer. Those properties play an important rule in finding a normal form for $\mathcal{S}_{(\ell,m,n)}$ in this case.

Lemma 4.4.2 *The following relations hold in* $\mathcal{S}_{(\ell,m,n)}$ *:*

1. $ab^{2t} = ab^m,\ ac^{2t} = ac^n$,

2. $ab^{2t+1} = ab,\ ac^{2t+1} = ac$.

Proof. Since $(\ell - 1)$ is even, Part (3) of Lemma 4.4.1 implies that

$$ab^i a^{\ell-1} = a^\ell b^i. \tag{4.14}$$

Multiplying both sides of (4.14) by a from the right gives:

$$ab^i a^\ell = a^\ell b^i a = ab^{m-i}. \tag{4.15}$$

By Lemma 4.4.1(1)

$$ab^i a^\ell = ab^i. \tag{4.16}$$

Thus (4.15) and (4.16) imply that $ab^i = ab^{m-i}$. By taking $i = 1$ we get $ab = ab^{m-1}$ which implies that $ab^2 = ab^m$ and therefore

$$ab^3 = ab,\ ab^4 = ab^2 = ab^m,\ ab^5 = ab,\ ab^6 = ab^2 = ab^m, \ldots.$$

Hence,

$$\forall t \in \mathbf{N}\ (ab^{2t} = ab^m,\ ab^{2t+1} = ab).$$

Similarly, one can show that $ac^{2t} = ac^n$ and $ac^{2t+1} = ac$. ■

Corollary 4.4.2 *If* a, b *and* c *are as in the presentation of* $\mathcal{S}_{(\ell,m,n)}$ *then*

1. $a^i b^{2t+1} c^{2d+1} b^{2r+1} = a^i bcb$,

2. $a^i b^{2t+1}c^{2d+1}b^{2r} = a^i bcb^m$,

3. $a^i b^{2t+1}c^{2d}b^{2r+1} = a^i bc^n b$,

4. $a^i b^{2t+1}c^{2d}b^{2r} = a^i bc^n b^m$,

5. $a^i b^{2t}c^{2d+1}b^{2r+1} = a^i b^m cb$,

6. $a^i b^{2t}c^{2d+1}b^{2t} = a^i b^m cb^m$,

7. $a^i b^{2t}c^{2d}b^{2r+1} = a^i b^m c^n b$,

8. $a^i b^{2t}c^{2d}b^{2r} = a^i b^m c^n b^m$.

Proof. Insert a^ℓ between each two letters and apply Lemmas 4.4.1 and 4.4.2.

In the next theorem we will see that $\mathcal{S}_{(\ell,m,n)}$ has different types of normal forms depending on whether m, n are odd, even or mixed.

Theorem 4.4.6 *If ℓ is odd, then we have four cases :*

(i) m and n are even and then :

$$N(\mathcal{S}_{(\ell,m,n)}) = \{\ b^j, b^i a^q, b^i a^q b^d, b^i c^j b^k, b^i a^q b^d c^r b^t, b^i c^j b^k a^q b^d, b^i c^j b^k a^q b^d c^r b^t :$$

$$0 \le i,k \le m, 1 \le j, \le n, 1 \le q \le \ell, 1 \le r \le 2, 0 \le d,t \le 2\ \};$$

(ii) m is odd and n is even, and then:

$$N(\mathcal{S}_{(\ell,m,n)}) = \{\ b^j, b^i a^q, b^i a^q b^d, b^i c^j b^k, b^i a^q b^d c^r b^t, b^i c^j b^k a^q b^d, b^i c^j b^k a^q b^d c^r b^t :$$

$$0 \le i,k \le m, 1 \le j, \le n, 1 \le q \le \ell, 1 \le r \le 2, 0 \le d,t \le 1\ \};$$

(iii) m is even and n is odd, and then :

$$N(\mathcal{S}_{(\ell,m,n)}) = \{\ a^q, b^i a^q, b^j, b^i a^q b^d, b^i c^j b^k, b^i a^q b^d c^r b^t, b^i c^j b^k a^q b^d, b^i c^j b^k a^q b^d c^r b^t :$$

$$0 \leq i, k \leq m, 1 \leq j, \leq n, 1 \leq q \leq \ell, 1 \leq r \leq 1, 0 \leq d, t \leq 2 \};$$

(iv) m and n are odd, and then :

$$N(\mathcal{S}_{(\ell,m,n)}) = \{a^q, b^i a^q, b^j, b^i a^q b^d, b^i c^j b^k, b^i a^q b^d c^r b^t, b^i c^j b^k a^q b^d, b^i c^j b^k a^q b^d c^r b^t :$$

$$0 \leq i, k \leq m, 1 \leq j, \leq n, 1 \leq q \leq \ell, 1 \leq r \leq 1, 0 \leq d, t \leq 1 \}.$$

Proof. (i) By applying Corollary 4.4.2 to the form given in Theorem 4.4.1 we obtain the above-mentioned form in this case.

(ii) Since m is odd, Lemma 4.4.2 implies that $ab^m = ab$. Also part (1) of Lemma 4.4.2 implies that $ab^m = ab^2$ i.e. $ab = ab^2$ which implies that

$$a^i b^j = a^i b \;\forall i, j : 1 \leq i \leq \ell,\; 1 \leq j \leq m. \tag{4.17}$$

Apply Lemma 4.4.2, Corollary 4.4.2 and (4.17) to the normal form $N(\mathcal{S}_{(\ell,m,n)})$, given in Theorem 4.4.1, to obtain the given form.

(iii) Since n is odd, Lemma 4.4.2 implies that $ac^n = ac$. and $ac^n = ac^2$ i.e. $ac = ac^2$ which implies that

$$a^i c^j = a^i c \;\forall\; i, j : 1 \leq i \leq \ell,\; 1 \leq j \leq m. \tag{4.18}$$

Again, apply Lemma 4.4.2, Corollary 4.4.2 and (4.18) to $N(\mathcal{S}_{(\ell,m,n)})$ to obtain the given form.

(iv) Since m and n are odd, $a^i b^j = a^i b$ for all $1 \leq j \leq n, 1 \leq i \leq \ell$, and $a^i c^j = a^i c$ for all $1 \leq j \leq m, 1 \leq i \leq \ell$. Now again apply Lemma 4.4.2, and Corollary 4.4.2 to $N(\mathcal{S}_{(\ell,m,n)})$ to obtain the given form. ■

The following theorem gives the order of $\mathcal{S}_{(\ell,m,n)}$ for each of the above cases.

Theorem 4.4.7 *The order of the semigroup $\mathcal{S}_{(\ell,m,n)}$ where ℓ is odd, is as given in the following table:*

m	n	$\lvert\mathcal{S}_{(\ell,m,n)}\rvert$
even	*even*	$(m+1)^2(21\ell n+n)+21\ell(m+1)+m$
even	*odd*	$(m+1)^2(21\ell n+n)+12\ell(m+1)+m$
odd	*even*	$(m+1)^2(10\ell n+n)+10\ell(m+1)+m$
odd	*odd*	$(m+1)^2(6\ell n+n)+6\ell(m+1)+m$

Table 4.6.

Proof. The results follow by counting the elements in the corresponding normal forms of Theorem 4.4.6. ■

Remark 4.4.2 Note that the effect of ℓ being odd on the normal form $N(\mathcal{S}_{(\ell,m,n)})$ of Theorem 4.4.1 is determined by Lemma 4.4.2 and Corollary 4.4.2.

Next, we examine the set of idempotents of $\mathcal{S}_{(\ell,m,n)}$ when ℓ is odd.

4.4.3 The idempotents of $\mathcal{S}_{(\ell,m,n)} : \ell$ is odd.

In this part we find all the idempotents of $\mathcal{S}_{(\ell,m,n)}$ for the case ℓ is odd.

Theorem 4.4.8 *Let ℓ be an odd integer; and let $1 \le i,k \le m,\ 1 \le j \le n$, $1 \le f \le 2$. Then, the sets of the idempotents in the semigroups $\mathcal{S}_{(\ell,m,n)}$ are as displayed in the following tables :*

1	a^ℓ	11	$b^i c^j a^\ell b^{i \pmod 2} c^{j \pmod 2}$
2	$a^\ell b^2$	12	$b^i c^j a^\ell b^f c^{j \pmod 2} b^{(i+f) \pmod 2}$
3	$a^\ell b^2 c^2$	13	$b^i c^n b^{m-i}$
4	$a^\ell b^f c^2 b^f$	14	$b^i c^j b^k a^\ell b^{((i+k) \pmod 2)} c^{j \pmod 2}$
5	$a^\ell c^2$	15	$b^i c^j b^k a^\ell b^f c^{j \pmod 2} b^{(i+k+f) \pmod 2}$
6	b^m	16	c^n
7	$b^i a^\ell b^{i \pmod 2}$	17	$c^j a^\ell b^2 c^{j \pmod 2}$
8	$b^i a^\ell b^{i \pmod 2} c^2$	18	$c^j a^\ell b^f c^{j \pmod 2} b^f$
9	$b^i a^\ell b^f c^2 b^{(i+f) \pmod 2}$	19	$c^i a^\ell c^{i \pmod 2}$
10	$b^m c^n$	20	$c^j b^i a^\ell b^{i \pmod 2} c^{j \pmod 2}$
		21	$c^j b^i a^\ell b^f c^{j \pmod 2} b^{(i+f) \pmod 2}$

m and n even.

1	a^ℓ	11	$b^i c^j a^\ell b^{i \pmod 2} c$
2	$a^\ell b^2$	12	$b^i c^j a^\ell b^f c b^{(i+f) \pmod 2}$
3	$a^\ell b^2 c$	13	$b^i c^n b^{m-i}$
4	$a^\ell b^f c b^f$	14	$b^i c^j b^k a^\ell b^{((i+k) \pmod 2)} c$
5	$a^\ell c$	15	$b^i c^j b^k a^\ell b^f c b^{(i+k+f) \pmod 2}$
6	b^m	16	c^n
7	$b^i a^\ell b^{i \pmod 2}$	17	$c^j a^\ell b^2 c$
8	$b^i a^\ell b^{i \pmod 2} c$	18	$c^j a^\ell b^f c b^f$
9	$b^i a^\ell b^f c b^{(i+f) \pmod 2}$	19	$c^i a^\ell c$
10	$b^m c^n$	20	$c^j b^i a^\ell b^{i \pmod 2} c$
		21	$c^j b^i a^\ell b^f c b^{(i+f) \pmod 2}$

m even, n odd.

1	a^ℓ	11	$b^i c^j a^\ell b c^{j \pmod 2}$
2	$a^\ell b$	12	$b^i c^j a^\ell b c^{j \pmod 2} b$
3	$a^\ell b c^2$	13	$b^i c^n b^{m-i}$
4	$a^\ell b c^2 b$	14	$b^i c^j b^k a^\ell b c^{j \pmod 2}$
5	$a^\ell c^2$	15	$b^i c^j b^k a^\ell b c^{j \pmod 2} b$
6	b^m	16	c^n
7	$b^i a^\ell b$	17	$c^j a^\ell b c^{j \pmod 2}$
8	$b^i a^\ell b c^2$	18	$c^j a^\ell b c^{j \pmod 2} b$
9	$b^i a^\ell b c^2 b$	19	$c^i a^\ell c^{i \pmod 2}$
10	$b^m c^n$	20	$c^j b^i a^\ell b c^{j \pmod 2}$
		21	$c^j b^i a^\ell b c^{j \pmod 2} b$

m odd, n even.

1	a^ℓ	11	$b^i c^j a^\ell bc$
2	$a^\ell b$	12	$b^i c^j a^\ell bcb$
3	$a^\ell bc^2$	13	$b^i c^n b^{m-i}$
4	$a^\ell bcb$	14	$b^i c^j b^k a^\ell bc$
5	$a^\ell c$	15	$b^i c^j b^k a^\ell bcb$
6	b^m	16	c^n
7	$b^i a^\ell b$	17	$c^j a^\ell bc$
8	$b^i a^\ell bc$	18	$c^j a^\ell bcb$
9	$b^i a^\ell bcb$	19	$c^i a^\ell c$
10	$b^m c^n$	20	$c^j b^i a^\ell bc$
		21	$c^j b^i a^\ell bcb$

Both m and n odd.

As before, let $E(S_{(\ell,m,n)})$ stands for the set of idempotents in the semigroup $S_{(\ell,m,n)}$. Then, by counting the number of elements in each of the above tables, one can easily find the order of the set of idempotents in $S_{(\ell,m,n)}$ as given in the following table:

m	n	$\lvert E(S_{(\ell,m,n)})\rvert$
even	even	$n(3(m+1)^2+1)+4(m+2)+m+1$
even	odd	
odd	odd	$n(2(m+1)^2+1)+4(m+2)$
odd	even	

4.4.4 The maximal subgroups of $S_{(\ell.m.n)} : \ell$ is odd.

One can use the same technique we used in the previous section to prove that most of the maximal subgroups of $S_{(\ell,m,n)}$ in this case are cyclic as described in the next theorem.

Theorem 4.4.9 *Let ℓ be an odd integer. Then,*

1. *if m and n are odd, all the maximal subgroups of $S_{(\ell,m,n)}$ are cyclic except the groups $G_{b^m c^n}$, $G_{b^i c^n b^{m-i}} : 1 \le i \le m$,*
2. *if m is even and n is odd, all the maximal subgroups of $S_{(\ell,m,n)}$ are cyclic except the groups $G_{b^m c^n}$, $G_{b^i c^n b^{m-i}} : 1 \le i \le m$,*
3. *if m is odd and n is even, all the maximal subgroups of $S_{(\ell,m,n)}$ are cyclic except the groups $G_{b^m c^n}, G_{b^i c^n b^{m-i}} : 1 \le i \le n$,*
4. *if m and n are even, all the maximal subgroups of $S_{(\ell,m,n)}$ are as in Theorem 4.4.5 with the powers of b and c reduced modulo 2 whenever they occur on the right-hand side of a.*

Proof. 1. Follows by applying (4.17) and (4.18) to Table 4.5.
2. Follows by applying Lemma 4.4.2. Corollary 4.4.2 and (4.18) to Table 4.5.
3. Follows by applying Lemma 4.4.2, Corollary 4.4.2 and (4.17) to Table 4.5.
4. Follows by applying Lemma 4.4.2 and Corollary 4.4.2 to Table 4.5. ∎

Chapter 5

The $\mathcal{D}$-classes structure of $\mathcal{S}_{(\ell,m,n)}$

In this chapter we investigate the $\mathcal{D}$-classes of $S_{(\ell,m,n)}$. The chapter goes on to relate the $\mathcal{D}$-classes of $S_{(\ell,m,n)}$ by means of "immediately above", and ends with a diagram relating the $\mathcal{D}$-classes.

5.1 The $\mathcal{D}$-classes of $S_{(\ell,m,n)}$

Since $S_{(\ell,m,n)}$ is finite, $\mathcal{D}_x = J_x$ for all $x \in S_{(\ell,m,n)}$.

Lemma 5.1.1 *Let $w_1 \in \{\ b,\ c\ \}^+ \cap\ N(S_{(\ell,m,n)})$, and let $w_2 = b^i c^j b^k$: $1 \le i \le m,\ 0 \le j \le n, 0 \le k \le m$. Then $w_1 a^d w_2 \in \mathcal{D}_{w_1 a w_2}$ for all d : $1 \le d \le \ell$.*

Proof. Let $d \in \mathbf{N}$ be arbitrary chosen. Then Lemma 4.4.1 implies that

$$w_1 a^d w_2 = \begin{cases} w_1 a w_2 a^{d-1} : d \text{ odd}, \\ w_1 a b^{m-i} c^{n-j} b^{m-k} a^{d-1} : d \text{ even}. \end{cases}$$

If d is even, then

$$w_1 a^d w_2 = w_1 a b^{m-i} c^{n-j} b^{m-k} a^{d-1} = w_1 a b^i c^j b^k (b^{m-k-2i} c^{n-2j} b^{m-k}) a^{d-1}$$

$$= w_1aw_2(b^{m-k-2i}c^{n-2j}b^{m-k})a^{d-1}.$$

Hence, $w_1a^dw_2 = w_1aw_2u$ for some $u \in \mathcal{S}^1_{(\ell,m,n)}$. Similarly, one can show that $w_1aw_2 = w_1a^dw_2v$ for some $v \in \mathcal{S}^1_{(\ell,m,n)}$. This shows that w_1aw_2 and $w_1a^dw_2$ are $\mathcal{J}$-related and thus $\mathcal{D}$-related. ∎

The combined effect of Lemma 4.4.1 and Lemma 5.1.1 is as follows:

Lemma 5.1.2 *If $w_1, w_2 \in \{\ b,\ c\ \}^+ \cap N(S_{(\ell,m,n)})$, then*

$$w_1a^dw_2 \in \mathcal{D}_{bcba^2bcb} \Longleftrightarrow w_2 = b^ic^jb^k : 1 \leq i,j; 0 \leq k.$$

Proof. Let $u = w_1a^2w_2$. By Lemma 5.1.1, it is enough to prove the lemma for u. Suppose that $u \in \mathcal{D}_{bcba^2bcb}$, so that there exist x, y in $\mathcal{S}^1$ such that $u = xbcba^2bcby$. Note that whatever x and y, there exist integers $d, r : 1 \leq d, r$ such that b^dc^r appears in the immediate right-hand side of a^2. It then follows by Theorems 4.4.1, 4.4.2 and 4.4.6 that $w_2 = b^ic^jb^k$ for some integers $i, j, k : 1 \leq i, j; 0 \leq k$.

Conversely, if $w_2 = b^ic^jb^k : 1 \leq i, j;\ 0 \leq k$ then

$$au = a^3w_1w_2 \Longrightarrow \exists w_3 \in \mathcal{S}^1_{(\ell,m,n)} : auw_3 = a^3bcb \text{ (Lemma 4.4.1)}$$

$$\Longrightarrow a^{\ell-1}uw_3 = abcb \text{ (Lemma 4.4.1)}. \tag{5.1}$$

Multiplying both sides of (5.1) by $bcba$ from the left gives:

$$bcba^{\ell}uw_3 = bcba^2bcb. \tag{5.2}$$

Similarly, one can show that there exist $u_1, u_2 \in \mathcal{S}^1_{(\ell,m,n)}$ such that

$$u_1bcba^2bcbu_2 = u. \tag{5.3}$$

It then follows from (5.2) and (5.3) that u and $bcba^2bcb$ are $\mathcal{J}$-related and thus $\mathcal{D}$-related. ∎

We now immediately deduce

Lemma 5.1.3 *The $\mathcal{D}$-class of bcbabcb is:*

$$\mathcal{D}_{bcbabcb} = \{\ b^i c^j b^k a^d b^r c^t b^e : 1 \le d, r, t; 0 \le i, j, k, e\ \}.$$

By counting the elements in $\mathcal{D}_{bcbabcb}$ one can deduce

Corollary 5.1.1 *The order of the $\mathcal{D}$-class of bcbabcb is as given by the following formula:*

$$|\mathcal{D}_{bcbabcb}| = \begin{cases} \ell mn[(m+1)^2 + n(m+1)^3] : \ell \text{ is even}, \\ 12\ell(1+m+n(m+1)^2) : \ \ell \text{ is odd, } m, n \text{ are even}, \\ 2\ell(1+m+n(m+1)^2) : \ \ell, m, n \text{ are odd}, \\ 6\ell(1+m+n(m+1)^2) : \ \ell, n \text{ are odd, } m \text{ is even}, \\ 4\ell(1+m+n(m+1)^2) : \ \ell, m \text{ are odd, } n \text{ is even}. \end{cases}$$

Clearly, the complement of $\mathcal{D}_{bcbabcb}$ with respect to $N(S_{(\ell,m,n)})$ is:

$$(\mathcal{D}_{bcbabcb})^c = \{\ b^i c^j b^k a^d c^t b^e : 0 \le i, j, k, t, e, d\ \}.$$

Lemma 5.1.2 is not applicable to the other $\mathcal{D}$-classes. In fact $\mathcal{D}_{bcbabcb}$ is of a special type as we will see that *bcbabcb* represent the minimal two-sided ideal. Thus, we will find the other $\mathcal{D}$-classes with the help of the following technical lemmas. First, we need the following definition:

Definition 5.1.1 Let x, y be elements in a semigroup $\mathcal{S}$. Then x is called a divisor of y if and only if there exist $u, v \in \mathcal{S}^1$ such that $y = uxv$.

We first consider the case in which ℓ is even.

5.1.1 When ℓ is even

In this part ℓ stand for an even positive number: $\ell \ge 2$.

Lemma 5.1.4 *Let $u = bcb^p ac^r b : p, r \ge 0$, and let $v = w_1 a^d w_2 : w_1, w_2 \in N(S_{(\ell,m,n)}) \cap \{b, c\}^+$. Then u and v are $\mathcal{D}$-related if and only if the following conditions hold:*

1. $w_1 = b^i c^j b^p : i, j \geq 1$,

2. $w_2 = \begin{cases} c^r b^t,\ t \geq 1 : d \ \ \textit{odd, or} \ \ r = 0, \\ c^{n-r} b^t, t \geq 1 : d \ \ \textit{even.} \end{cases}$

Proof. Suppose that u and v are $\mathcal{D}$-related. By Lemma 5.1.3 $w_2 = c^q b^t$ for some $q, t : q, t \geq 0$. Since $u \ \mathcal{D} \ v$, it follows that

$$\exists x, y \in \mathcal{S}^1_{(\ell,m,n)} : xbcb^p ac^r by = w_1 a^d c^q b^t. \tag{5.4}$$

It then follows from (5.4), Lemmas 4.4.1, 4.4.2 and Theorems 4.4.2, 4.4.6 that a is not a divisor of x and c is not a divisor of y and thus

$$w_1 = b^i c^j b^p : i, j \geq 1.$$

Also, a^{d-1} must be a divisor of y. Hence, Lemmas 4.4.1 and 4.4.2 imply that

$$q = \begin{cases} r : d - 1 \ \text{ even, or } \ \ r = 0, \\ n - r : d - 1 \ \text{ odd.} \end{cases}$$

It, also, follows from (5.4) that $t \geq 1$. Hence, Conditions 1 and 2 hold. Conversely, if Conditions 1 and 2 hold, then

$$v = \begin{cases} b^i c^j b^p a^d c^r b^t : d \ \text{ odd, or } r = 0, \\ b^i c^j b^p a^d c^{n-r} b^t : d \ \text{ even.} \end{cases}$$

If d is odd, then choose $x = b^{m-i+1} c^{m-j+1}$, $y = a^{\ell-d+1} b^{m-t+1}$, $x' = b^{m+i-1} c^{m-+j-1}$, and $y' = a^{\ell+d-1} b^{m+t-1}$ to get

$$xvy = bcb^p ac^r b, \ x'(b^i c^j b^p a^d c^r b^t) y' = v.$$

Similarly, when d is even, choose x and x' as above; and $y = a^{\ell+d-1} b^{1+t}$, $y' = a^{\ell+1-d} b^{1+t}$ to get

$$xvy = bcb^p ac^r b, \ x'(b^i c^j b^p a^d c^r b^t) y' = v.$$

Hence, $u \ \mathcal{D} \ v$. ■

We now immediately deduce

Corollary 5.1.2 *Let p, q be fixed natural numbers. Then,*

1. $\mathcal{D}_{bcab} = \{\ b^ic^ja^db^t : 1 \le i,t \le m, 1 \le j \le n, 1 \le d \le \ell\}$;
2. $\mathcal{D}_{bcb^pab} = \{\ b^ic^jb^pa^db^t : 1 \le i,t \le m, 1 \le j \le n, 1 \le d \le \ell\ \}$;
3. $$\mathcal{D}_{bcac^qb} = \bigcup_{d\ odd} \{\ b^ic^ja^dc^qb^t : 1 \le i,t \le m, 1 \le d \le \ell, 1 \le j \le n\ \} \bigcup_{d\ even} \{\ b^ic^ja^dc^{n-q}b^t : 1 \le i,t \le m, 1 \le d \le \ell, 1 \le j \le n\ \};$$
4. $$\mathcal{D}_{bcb^pac^qb} = \bigcup_{d\ odd} \{\ b^ic^jb^pa^dc^qb^t : 1 \le i,t \le m, 1 \le d \le \ell, 1 \le j \le n\ \} \bigcup_{d\ even} \{\ b^ic^jb^pa^dc^{n-q}b^t : 1 \le i,t \le m, 1 \le d \le \ell, 1 \le j \le n\ \}.$$

Lemma 5.1.5

$$\mathcal{D}_{bac^qb} = \bigcup_{d\ odd} \{\ b^ia^dc^qb^t : 1 \le i,t \le m, 1 \le d \le \ell\ \} \bigcup_{d\ even} \{\ b^ia^dc^{n-q}b^t : 1 \le i,t \le m, 1 \le d \le \ell\ \}.$$

Proof. The Proof is similar to the proof of Lemma 5.1.4.

Lemma 5.1.6 1. $\mathcal{D}_{bcb^pac} = \{\ b^ic^jb^pa^dc^t : 1 \le i,j,d,t\ \}$,

2. $\mathcal{D}_{bcac} = \{\ b^ic^ja^dc^t : 1 \le i,j,d,t\ \}$.

Proof. 1. Follows from Lemma 5.1.3 and Corollary 5.1.2 (2,4).
2. Follows from Lemma 5.1.3 and Corollary 5.1.2 (1,3). ∎

Lemma 5.1.7 *Let $u = cb^pac^qb : p,q \ge 1$. Then,*

$$\mathcal{D}_u = \bigcup_{d\ odd} \{\ c^jb^pa^dc^qb^t : 1 \le j,t \le m, 1 \le d \le \ell\ \} \bigcup_{d\ even} \{\ c^jb^pa^dc^{n-q}b^t : 1 \le j,t \le m, 1 \le d \le \ell\ \}.$$

Proof. The proof is similar to the proof of Lemma 5.1.4. ∎

Lemma 5.1.8 *Let $u = cb^pab : p \geq 0$, and let $v = w_1a^dw_2 : w_1, w_2 \in N(\mathcal{S}_{(\ell,m,n)}) \cap \{b,c\}^+$. Then, u and v are $\mathcal{D}$-related if and only if $w_1 = c^jb^p$ and $w_2 = b^t : j,t \geq 1$.*

Proof. Assume that u and v are $\mathcal{D}$-related. It follows from Lemma 5.1.3 and Lemma 5.1.6 that $w_2 = b^t$ or c^r for some $t, r \geq 1$. Assume that $w_2 = c^r$. Then,

$$\exists x, y \in \mathcal{S}^1_{(\ell,m,n)} : xcb^paby = w_1a^dc^r$$

which is impossible. Thus, $w_2 = b^t : t \geq 1$ and

$$\exists x, y \in \mathcal{S}^1_{(\ell,m,n)} : xcb^paby = w_1a^db^t. \tag{5.5}$$

It follows from (5.5) that a is not a divisor of x and c is not a divisor of y and thus $w_1 = b^ic^jb^p$ for some $i,j : 0 \leq i, 1 \leq j$. Also,

$$\exists x', y' \in \mathcal{S}^1_{(\ell,m,n)} : xb^ic^jb^pa^dc^ry = cb^pab$$

which implies that $i = 0$ and thus $w_1 = c^jb^p, w_2 = b^t : j,t \geq 1$. Conversely, if $w_1 = c^jb^p, w_2 = b^t : j,t \geq 1$, then one can easily show that $u \ \mathcal{D} \ v$. ∎

Corollary 5.1.3 *Let $p \geq 1$ be fixed. Then,*

1. $\mathcal{D}_{cb^pab} = \{ \ c^jb^pa^db^t : 1 \leq j \leq n, \leq d \leq \ell, 1 \leq t \leq m \ \}$,
2. $\mathcal{D}_{cab} = \{ \ c^ja^db^t : 1 \leq j \leq n, \leq d \leq \ell, 1 \leq t \leq m \ \}$.

Lemma 5.1.9 $\mathcal{D}_{cb^pac} = \{ \ c^jb^pa^dc^r : 1 \leq j, r \leq n, 1 \leq d \leq \ell\}$.

Proof. The result follows from Lemma 5.1.6 and Corollary 5.1.3 (1). ∎

Lemma 5.1.10

$$\mathcal{D}_{cacb} = \{\ c^j a^d c^r b^t : 0 \leq j \leq n, 1 \leq r \leq n, 1 \leq d \leq \ell, 1 \leq t \leq m\}.$$

Proof. Let $X = \{\ c^j a^d c^r b^t : 1 \leq j, r \leq n, 1 \leq d \leq \ell, 1 \leq t \leq m\}$, and let $w \in \mathcal{D}_{cacb}$. By Lemma 5.1.3 $w = w_1 a^d c^r b^t$ for some $r, t : r, t \geq 0$ and $w_1 \in \mathcal{S}_{(\ell,m,n)} \cap \{b, c\}^+$. Thus,

$$\exists x, y \in \mathcal{S}^1_{(\ell,m,n)} : xcacby = w_1 a^d c^r b^t. \tag{5.6}$$

It follows from (5.6) that c is not a divisor of y and thus $r, t \geq 1$. Assume that $w_1 = b^i c^j b^k : i, j, k \geq 0$. Since there exist $x', y' \in \mathcal{S}^1_{(\ell,m,n)}$ such that $cacb = x' w_1 a^d c^r b^t y'$ and c is not a divisor of y', it follows that $i = k = 0$ and thus $w = c^j a^d c^r b^t \in X$. Conversely, if $u \in X$, then one can easily show that $u\ \mathcal{D}\ cacb$. ■

Similarly, one can show the following lemma.

Lemma 5.1.11

1. $\mathcal{D}_{cac} = \{\ c^j a^d c^r : 0 \leq j \leq n, 1 \leq r \leq n, 1 \leq d \leq \ell\ \}$,
2. $\mathcal{D}_{bac} = \{\ b^j a^d c^r : 1 \leq j \leq m, 1 \leq r \leq n, 1 \leq d \leq \ell\ \}$,
3. $\mathcal{D}_{bab} = \{\ b^j a^d b^r : 0 \leq j \leq m, 1 \leq r \leq m, 1 \leq d \leq \ell\ \}$

Lemma 5.1.12 *Let $u = bcb^p a : p \geq 0$, and let $v \in \mathcal{S}_{(\ell,m,n)}$. Then,*

$$u\ \mathcal{D}\ v \iff v = b^i c^j b^p a^d : 1 \leq i \leq m, 1 \leq j \leq n, 1 \leq d \leq \ell.$$

Proof. Assume that $u\,\mathcal{D}\,v$. It follows from Corollary 5.1.2 (2,4), Lemma 5.1.6 (1) and Lemma 5.1.3 that $v = b^i c^j b^k a^d$ for some $i, j, k, d : 0 \leq i, j, k; 1 \leq d$. Since $u\ \mathcal{D}\ v$, it follows that

$$\exists x, y : x b^i c^j b^k a^d y = bcb^p a, \tag{5.7}$$

$$\exists x', y' : b^i c^j b^k a^d = x' bcb^p a y'. \tag{5.8}$$

It then follows from (5.7) and (5.8) that a is not a divisor of x or x' and thus $k = p$. Conversely, if $v = b^i c^j b^p a^d$, then one can easily show that $u \;\mathcal{D}\; v$. ∎

Corollary 5.1.4 *Let $p \in \mathbf{N}$ be fixed. Then,*

1. $\mathcal{D}_{bcb^pa} = \{\ b^i c^j b^p a^d : 1 \le i \le m, 1 \le j \le n, 1 \le d \le \ell\ \}$,
2. $\mathcal{D}_{bca} = \{\ b^i c^j a^d : 1 \le i \le m, 1 \le j \le n, 1 \le d \le \ell\ \}$.

Similarly, one can prove the next lemma.

Lemma 5.1.13

1. $\mathcal{D}_a = \{\ a^i : 1 \le i \le \ell\}$;
2. $\mathcal{D}_{cb^pa} = \{\ c^j b^p a^d : 1 \le j \le n, 1 \le d \le \ell\ \}$,
3. $\mathcal{D}_{ca} = \{\ c^j a^d : 1 \le j \le n, 1 \le d \le \ell\ \}$,
4. $\mathcal{D}_{ba} = \{\ b^j a^d : 1 \le j \le m, 1 \le d \le \ell\ \}$.

The next lemma which gives the remaining $\mathcal{D}$-classes is a consequence of Lemma 4.4.1 and the definition of the $\mathcal{J}$-classes.

Lemma 5.1.14 *The $\mathcal{D}$-classes contained in $S_{(\ell,m,n)} \cap \{b,c\}^+$ are :*

1. $\mathcal{D}_b = \{\ b^i : 1 \le i \le m\}$;
2. $\mathcal{D}_c = \{\ c^j : 1 \le j \le n\}$;
3. $\mathcal{D}_{bcb} = \{\ b^i c^j b^k : 0 \le k \le m,\ 1 \le i \le m, 1 \le j \le n\}$;
4. $\mathcal{D}_{cb} = \{\ c^j b^i : 1 \le i \le m, 1 \le j \le n\ \}$.

The following lemma gives a set of representatives for the $\mathcal{D}$-classes in $S_{(\ell,m,n)}$.

Lemma 5.1.15 *Let $x \in \mathcal{S}_{(\ell,m,n)}$, $1 \le p \le m, 1 \le q \le n$, and let*

$$X = \{\ bcbabcb,\ bcab,\ bcb^pab,\ bcac^qb,\ bcb^pac^qb,\ bac^qb,\ bcb^pac,\ bcac,\ cb^pac^qb,\ cb^pab,\ cab,\ cb^pac,\ cacb,\ cac,\ cab,\ bac,\ bab,\ bcb^pa,\ bca, cb^pa,\ ca,\ ba,\ a,\ b,\ c,\ cb\ \}.$$

Then there exists $y \in X$ such that $x \in \mathcal{D}_y \quad (\mathcal{D}_x = \mathcal{D}_y)$.

Proof. The proof follows by Theorem 4.4.2 and Lemmas 5.1.3, 5.1.5, 5.1.6, 5.1.7, 5.1.9, 5.1.10, 5.1.11, 5.1.13 and 5.1.14; Corollaries 5.1.2, 5.1.3 and 5.1.4. ■

5.1.2 When ℓ is odd

In the case of ℓ being odd, we apply Lemma 4.4.2, Corollary 4.4.2, (4.17) and (4.18) to the $\mathcal{D}$-classes obtained in the case ℓ is even to deduce the $\mathcal{D}$-classes of $\mathcal{S}_{(\ell,m,n)}$. As an example, we state the $\mathcal{D}$-classes of $\mathcal{S}_{(\ell,m,n)}$, when ℓ, n are odd and m is even, in the following proposition.

Proposition 5.1.1 *Let ℓ and n be odd positive integers, let m be an even positive integer, and let p be a fixed positive integer. Then, the $\mathcal{D}$-classes of $\mathcal{S}_{(\ell,m,n)}$ are:*

1. $\mathcal{D}_{bcbabcb} = \{b^ic^jb^ka^qb^rcb^d : 0 \le i, k \le m, 0 \le j \le n, 1 \le q \le \ell, 1 \le r \le 2, 0 \le d \le 2\}$;
2. $\mathcal{D}_{bcab} = \{\ b^ic^ja^db^t : 1 \le i \le m, 1 \le j \le n, 1 \le d \le \ell, 1 \le t \le 2\ \}$;
3. $\mathcal{D}_{bcb^pab} = \{\ b^ic^jb^pa^db^t : 1 \le i \le m, 1 \le j \le n, 1 \le d \le \ell, 1 \le t \le 2\ \}$;
4. $\mathcal{D}_{bcacb} = \{\ b^ic^ja^dcb^t : 1 \le i \le m, 1 \le t \le 2, 1 \le d \le \ell, 1 \le j \le n\ \}$;

5. $\mathcal{D}_{bcb^pacb} = \{\ b^ic^jb^pa^dcb^t : 1 \le i \le m, 1 \le t \le 2, 1 \le d \le \ell, 1 \le j \le n\ \}$;
6. $\mathcal{D}_{bacb} = \{\ b^ia^dcb^t : 1 \le i \le m, 1 \le t \le 2, 1 \le d \le \ell\ \}$;
7. $\mathcal{D}_{bcb^pac} = \{\ b^ic^jb^pa^dc : 1 \le i \le m, 1 \le j \le n, 1 \le d \le \ell\ \}$;
8. $\mathcal{D}_{bcac} = \{\ b^ic^ja^dc : 1 \le i \le m, 1 \le j \le n, 1 \le d \le \ell\ \}$;
9. $\mathcal{D}_{cb^pacb} = \{\ c^jb^pa^dcb^t : 1 \le j \le m, 1 \le d \le \ell\ \}$;
10. $\mathcal{D}_{cb^pab} = \{\ c^jb^pa^db^t : 1 \le j \le n, \le d \le \ell, 1 \le t \le 2\ \}$;
11. $\mathcal{D}_{cab} = \{\ c^ja^db^t : 1 \le j \le n, \le d \le \ell, 1 \le t \le 2\ \ \}$;
12. $\mathcal{D}_{cb^pac} = \{\ c^jb^pa^dc : 1 \le j \le n, 1 \le d \le \ell\}$;
13. $\mathcal{D}_{cacb} = \{\ c^ja^dcb^t : 1 \le j, r \le n, 1 \le d \le \ell, 1 \le t \le 2\}$;
14. $\mathcal{D}_{cac} = \{\ c^ja^dc : 0 \le j \le n, 1 \le d \le \ell\ \}$;
15. $\mathcal{D}_{bac} = \{\ b^ja^dc : 1 \le j \le m, 1 \le d \le \ell\ \}$;
16. $\mathcal{D}_{bab} = \{\ b^ja^db^r : 0 \le j \le n, 1 \le r \le 2, 1 \le d \le \ell\ \}$;
17. $\mathcal{D}_{bcb^pa} = \{\ b^ic^jb^pa^d : 1 \le i \le m, 1 \le j \le n, 1 \le d \le \ell\ \}$;
18. $\mathcal{D}_{bca} = \{\ b^ic^ja^d : 1 \le i \le m, 1 \le j \le n, 1 \le d \le \ell\ \}$;
19. $\mathcal{D}_{cb^pa} = \{\ c^jb^pa^d : 1 \le j \le n, 1 \le d \le \ell\ \}$;
20. $\mathcal{D}_{ca} = \{\ c^ja^d : 1 \le j \le n, 1 \le d \le \ell\ \}$;
21. $\mathcal{D}_{ba} = \{\ b^ja^d : 1 \le j \le m, 1 \le d \le \ell\ \}$;
22. $\mathcal{D}_a = \{\ a^i : 1 \le i \le \ell\}$;
23. $\mathcal{D}_b = \{\ b^i : 1 \le i \le m\}$;
24. $\mathcal{D}_c = \{\ c^j : 1 \le j \le n\}$;
25. $\mathcal{D}_{bcb} = \{b^ic^jb^k : 0 \le i \le m, 1 \le j \le n, 1 \le k \le n\}$;
26. $\mathcal{D}_{cb} = \{\ c^jb^i : 1 \le i \le m, 1 \le j \le n\ \}$.

We close this section with a table that gives formulas for the number of $\mathcal{D}$-classes in $S_{(\ell,m,n)}$ according to the type of ℓ, m and n as even or odd. The formulas follow by counting the elements of X in Lemma 5.1.6 in the light of Lemma 4.4.2 and Corollary 4.4.2.

ℓ	m	n	number of $\mathcal{D}$-classes
even			$2(mn+3m+n+8)$
odd	even	even	$2(2m+3m+10)$
odd	odd	even	
odd	even	odd	$2(m+3m+9)$
odd	odd	odd	

In the next section we relate the $\mathcal{D}$-classes of $S_{(\ell,m,n)}$ by means of "immediately above".

5.2 Relating the $\mathcal{D}$-classes of $S_{(\ell,m,n)}$

Let $\mathcal{S}$ be a semigroup and $x, y, z \in \mathcal{S}$. Then $\mathcal{D}_x$ is said to be immediately above $\mathcal{D}_y$ if and only if $\mathcal{S}^1y\mathcal{S}^1 \subseteq \mathcal{S}^1x\mathcal{S}^1$ and if $\mathcal{S}^1y\mathcal{S}^1 \subseteq \mathcal{S}^1z\mathcal{S}^1 \subseteq \mathcal{S}^1x\mathcal{S}^1$ then $\mathcal{S}^1y\mathcal{S}^1 = \mathcal{S}^1z\mathcal{S}^1$ or $\mathcal{S}^1z\mathcal{S}^1 = \mathcal{S}^1x\mathcal{S}^1$.

Since $S_{(\ell,m,n)}$is finite, $\mathcal{D}_x = J_x$ for all $x \in S_{(\ell,m,n)}$. Thus to relate the $\mathcal{D}$-classes by means of "immediately above", it is enough to relate the $\mathcal{J}$-classes. Let us recall the definition of the *divisor,* given in Definition 5.1.1.

Definition 5.2.1 Let $\mathcal{S}$ be a semigroup, and let $x, y \in \mathcal{S}$. Then x is said to be a *divisor of y* if and only if there exist $u, v \in \mathcal{S}^1$ such that $y = uxv$.

The next proposition helps in relating the two-sided ideals by inclusion.

Proposition 5.2.1 *Let x, y be elements of a semigroup $\mathcal{S}$. Then;*

$$\mathcal{S}^1x\mathcal{S}^1 \subseteq \mathcal{S}^1y\mathcal{S}^1 \iff y \textit{ is a divisor of } x.$$

Proof. Suppose that y is a divisor of x, so that there exist $u, v \in \mathcal{S}^1$ such that $x = uyv$. If $z \in \mathcal{S}^1x\mathcal{S}^1$, then $z = sxs'$ for some $s, s' \in \mathcal{S}^1$. It then follows that $z = s(uyv)s' = su(y)vs'$ which establishes that $z \in \mathcal{S}^1y\mathcal{S}^1$ and hence $\mathcal{S}^1x\mathcal{S}^1 \subseteq \mathcal{S}^1y\mathcal{S}^1$.

Conversely, if $\mathcal{S}^1 x \mathcal{S}^1 \subseteq \mathcal{S}^1 y \mathcal{S}^1$, then $x = sys'$ for some $s, s' \in \mathcal{S}^1$ and thus y is a divisor of x. ■

We now immediately deduce

Corollary 5.2.1 *Let x be an element of a semigroup $\mathcal{S}$. Then $\mathcal{S}^1 x \mathcal{S}^1$ is the minimal two-sided ideal if and only if x is not a divisor of any $\mathcal{J}$-class representative and every representative of a $\mathcal{J}$-class is a divisor of x in $\mathcal{S}$.*

Corollary 5.2.2 *Let x, y, z be elements of a semigroup $\mathcal{S}$. Then $\mathcal{D}_x$ is immediately above $\mathcal{D}_y$ if and only if there exists $s \in \mathcal{D}_y$ such that x is a divisor of s and if z is a divisor of y then $z \in \mathcal{D}_x \cup \mathcal{D}_y$.*

The next lemma gives the minimal two-sided ideal in $S_{(\ell,m,n)}$.

Lemma 5.2.1 *The minimal two-sided ideal in $S_{(\ell,m,n)}$ is $\mathcal{S}^1_{(\ell,m,n)} bcbabcb \mathcal{S}^1_{(\ell,m,n)}$.*

Proof. By Lemma 4.4.1 and Lemma 5.1.15 *bcbabcb* is not a divisor of any of the representatives of the $\mathcal{D}$-classes (and thus the $\mathcal{J}$-classes) in $S_{(\ell,m,n)}$ other than $bcb^p abc^q b$. Since $bcb^p abc^q b \in J_{bcbabcb}$ and each $\mathcal{J}$-class representative is a divisor of an element in $J_{bcbabcb}$, it follows by Corollary 5.2.1 that $\mathcal{S}^1_{(\ell,m,n)} bcbabcb \mathcal{S}^1_{(\ell,m,n)}$ is the minimal two-sided ideal in $S_{(\ell,m,n)}$. ■

Definition 5.2.2 Let X be the set of the $\mathcal{D}$-classes representatives in a semigroup $\mathcal{S}$, and let $\emptyset \neq Y \subseteq X$. Then Y is called a *$\mathcal{D}$-classes layer representative set* if

$$(\forall x, y \in Y)\ \mathcal{S}^1 x \mathcal{S}^1 \subseteq \mathcal{S}^1 y \mathcal{S}^1 \quad \text{or} \quad \mathcal{S}^1 y \mathcal{S}^1 \subseteq \mathcal{S}^1 x \mathcal{S}^1 \quad \text{and}$$

$$\forall x \in X \setminus Y, \exists x' \in Y : \mathcal{S}^1 x \mathcal{S}^1 \subseteq \mathcal{S}^1 x' \mathcal{S}^1 \quad \text{or} \quad \mathcal{S}^1 x \mathcal{S}^1 \subseteq \mathcal{S}^1 x' \mathcal{S}^1.$$

The set of the $\mathcal{D}$-classes represented by Y is called a *$\mathcal{D}$-classes layer.*

The next lemma shows that the number of layers of the $\mathcal{D}$-classes in $S_{(\ell,m,n)}$ is independent of ℓ, m and n.

Lemma 5.2.2 *The semigroup $S_{(\ell,m,n)}$ has nine $\mathcal{D}$-classes layers.*

Proof. Let p, q be two fixed natural numbers, let X be as in Lemma 5.1.6, and let

- $Y_1 = \{a, b, c\}$, $L_1 = \{\ \mathcal{D}_a,\ \mathcal{D}_b,\ \mathcal{D}_c\ \}$.
- $Y_2 = \{cb, ba, ca\}$, $L_2 = \{\ \mathcal{D}_{cb},\ \mathcal{D}_{ba},\ \mathcal{D}_{ca}\}$;
- $Y_3 = \{cb^pa, bab, cac\}$, $L_3 = \{\ \mathcal{D}_{cb^pa},\ \mathcal{D}_{bab},\ \mathcal{D}_{cac}\ \}$;
- $Y_4 = \{cb^pab, bac, cab\}$, $L_4 = \{\ \mathcal{D}_{cb^pab},\ \mathcal{D}_{bac},\ \mathcal{D}_{cab}\ \}$;
- $Y_5 = \{cacb, bcb, cb^pac\}$, $L_5 = \{\ \mathcal{D}_{cacb},\ \ \mathcal{D}_{bcb}\ , \mathcal{D}_{cb^pac}\ \}$;
- $Y_6 = \{bcb^pa, bac^qb, bca\}$, $L_6 = \{\ \mathcal{D}_{bcb^pa},\ \mathcal{D}_{bac^qb},\ \mathcal{D}_{bca}\ \}$.
- $Y_1 = \{bcb^pac, cb^pac^qb, bcac, bcb^pab, bcab\}$,
  $L_7 = \{\ \mathcal{D}_{bcb^pac},\ \mathcal{D}_{cb^pac^qb},\ \mathcal{D}_{bcac}\ ,\ \mathcal{D}_{bcb^pab},\ \mathcal{D}_{bcab}\ \}$;
- $Y_1 = \{bcb^pac^qb, bcac^qb\}$, $L_8 = \{\ \mathcal{D}_{bcb^pac^qb},\ \mathcal{D}_{bcac^qb}\ \}$.

Clearly none of a, b or c is a divisor of another. Also, if $x \in X \setminus Y_1$ then either a, b or c is a divisor of x. Hence, L_1 is a $\mathcal{D}$-classes layer.

Similarly, L_2 is also a $\mathcal{D}$-classes layer.

By Lemma 4.4.1 none of cb^pa, bab or cac is a divisor of another. Also, if $x \in X \setminus (\cup_{i=1}^{3} Y_i)$ then one of them is a divisor of x. Thus, L_3 is a $\mathcal{D}$-classes layer.

In a similar way, one can show that L_4, L_5, L_6, L_7 and L_8 are $\mathcal{D}$-classes layers.

Lemma 5.1.15 implies that $(\cup_{i=1}^{8} L_i) \cup \{ \mathcal{D}_{bcbabcb} \}$ is the set of all $\mathcal{D}$-classes in $S_{(\ell,m,n)}$. This establishes that $S_{(\ell,m,n)}$ has nine $\mathcal{D}$-classes layers. ■

Remark 5.2.1 Note that the number of $\mathcal{D}$-classes in each of the layers $L_6, \cdots, L_8$ depends on whether ℓ is even or odd; and if ℓ is odd it then depends on whether n is even or odd. The following table gives the number of the $\mathcal{D}$-classes in each layer :

ℓ	n	L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8
even		3	3	$m+2$	$m+2$	$m+2$	$m+n+1$	$2m+mn+2$	$mn+n$
odd	even	3	3	$m+2$	$m+2$	$m+2$	$m+3$	$4m+2$	$2m+2$
odd	odd	3	3	$m+2$	$m+2$	$m+2$	$m+2$	$3m+2$	$m+1$

Using Proposition 5.2.1, one can easily deduce the following diagram in which the upper $\mathcal{D}$-class of any two connected $\mathcal{D}$-classes is immediately above the other one. As an example, we show that $\mathcal{D}_{bcb^pac^qb}$ is immediately above $\mathcal{D}_{bcbabcb}$.

Example 5.2.1 *The $\mathcal{D}$-class $\mathcal{D}_{bcb^pac^qb}$ is immediately above $\mathcal{D}_{bcbabcb}$.*

Proof. Let $x = bcb^pac^qb$. Then,

$$\begin{aligned} a^\ell x \underbrace{b^{p+1}c^{n+2-q}b} &= a(b^{m-1}c^{n-1}b^{m-p}c^q b \underbrace{b^{p+1}c^{n+2-q}b} \quad \text{(Lemma 4.4.1 (2))} \\ &= ab^{m-(1+p)}c^{n-1+q}b \underbrace{b^{p+1}c^{n+2-q}b} = abcb \text{ (Lemma 4.4.1 (7))} \\ &\Longrightarrow bcba^\ell x \underbrace{b^{p+1}c^{n+2-q}b} = bcbabcb. \end{aligned}$$

Thus bcb^pac^qb is a divisor of $bcbabcb$ and

$$\mathcal{S}^1_{(\ell,m,n)} bcbabcb \mathcal{S}^1_{(\ell,m,n)} \subseteq \mathcal{S}^1_{(\ell,m,n)} bcb^pac^qb \mathcal{S}^1_{(\ell,m,n)}.$$

Assume that there is $z \in \mathcal{S}^1$ such that

$$\mathcal{S}^1(bcbabcb)\mathcal{S}^1 \subseteq \mathcal{S}^1 z \mathcal{S}^1 \subseteq \mathcal{S}^1 bcb^pac^qb\mathcal{S}^1.$$

Then there exist $u, v, u', v' \in \mathcal{S}^1$ such that

$$z = ubcb^p ac^q bv, \quad bcbabcb = u'zv'.$$

Assume that $z \notin J_{bcbabcb}$. It then follows from Lemma 5.1.3 and the fact that $z = ubcb^p ac^q bv$ that

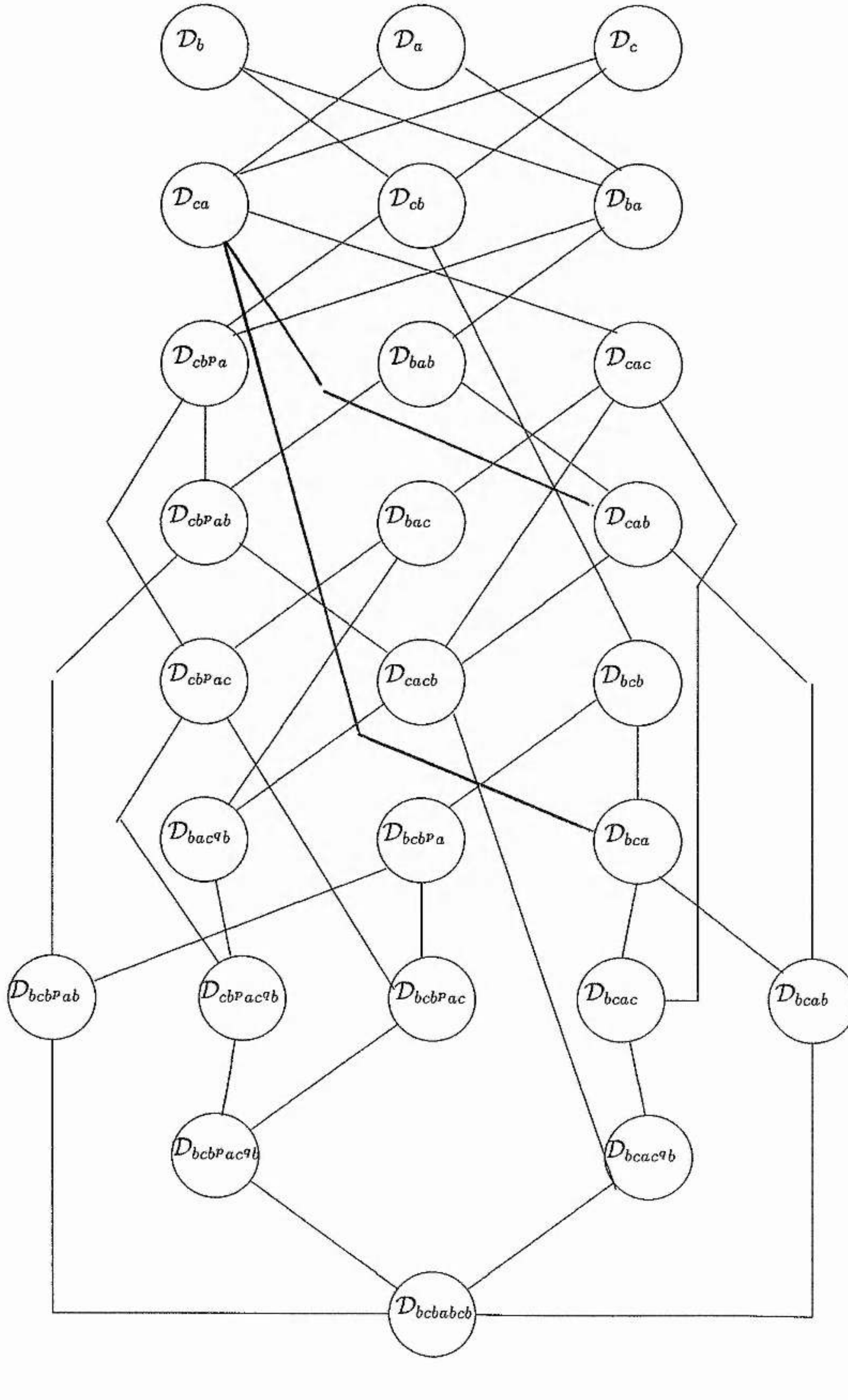
$$z = w_1 a^d c^r b^t : w_1 \in (N(\mathcal{S}_{(\ell,m,n)})) \cap \{b,c\}^+, \; r,t \geq 0, d \geq 1.$$

Thus $w_1 a^d c^r b^t = ubcb^p ac^q bv$ which implies that c is not a divisor of v and a is not a divisor of u and hence

$$w_1 a^d c^r b^t = b^i c^j b^p a^g c^q b^k : 1 \leq i,k \leq m, 1 \leq j \leq n, 1 \leq g \leq \ell. \tag{5.9}$$

It then follows from (5.9), Theorem 4.4.1 and Theorem 4.4.6 that $w_1 = b^i c^j b^k$, $i = d, r = q, k = t$ and Thus $z \in J_{bcb^p ac^q b}$. This shows that either $z \in J_{bcbabcb}$ or $z \in J_{bcb^p ac^q b}$. Hence, $\mathcal{D}_{bcb^p ac^q b}$ is immediately above $\mathcal{D}_{bcbabcb}$. ∎

We now use the same technique to deduce the following diagram.



$\mathcal{D}_b$
$\mathcal{D}_a$
$\mathcal{D}_c$
$\mathcal{D}_{ca}$
$\mathcal{D}_{cb}$
$\mathcal{D}_{ba}$
$\mathcal{D}_{cb^pa}$
$\mathcal{D}_{bab}$
$\mathcal{D}_{cac}$
$\mathcal{D}_{cb^pab}$
$\mathcal{D}_{bac}$
$\mathcal{D}_{cab}$
$\mathcal{D}_{cb^pac}$
$\mathcal{D}_{cacb}$
$\mathcal{D}_{bcb}$
$\mathcal{D}_{bac^qb}$
$\mathcal{D}_{bcb^pa}$
$\mathcal{D}_{bca}$
$\mathcal{D}_{bcb^pab}$
$\mathcal{D}_{cb^pac^qb}$
$\mathcal{D}_{bcb^pac}$
$\mathcal{D}_{bcac}$
$\mathcal{D}_{bcab}$
$\mathcal{D}_{bcb^pac^qb}$
$\mathcal{D}_{bcac^qb}$
$\mathcal{D}_{bcbabcb}$

Chapter 6

Schützenberger groups

6.1 Introduction

Each regular $\mathcal{D}$-class D of a semigroup $\mathcal{S}$ is associated with a group, the group of any of the isomorphic group $\mathcal{H}$-classes in D. In 1957, Schützenberger extended this result to all $\mathcal{D}$-classes of a semigroup $\mathcal{S}$. This group, which is called the *Schützenberger group* of the $\mathcal{D}$-class D, is constructed as follows: Let H be an $\mathcal{H}$-class in a semigroup $\mathcal{S}$, and let $h_1 \in H$, $x_1 \in \mathcal{S}^1$. It follows that if $h_1x_1 \in H$, then the inner translation $\rho_{x_1} : h \to hx_1$ defines a permutation on H. Let

$$T_r(H) = \{x \in S^1 : Hx = H\}, \quad T_\ell(H) = \{x \in S^1 : xH = H\}.$$

Then $T_r(H)$ $[T_\ell(H)]$, called the *right [left] stabilizer* of H, is a submonoid of $\mathcal{S}^1$. Let

$$P(H) = \{\rho_t|_H : t \in T_r(H)\}.$$

Using Green's Lemma, one can easily show that $P(H)$ is a group of permutations on H. In fact, $P(H)$ is a regular group, that is, for every $g_1, g_2 \in P(H)$, there is a unique $\rho \in P(H)$ such that $(g_1)\rho = g_2$.

Define an equivalence relation $\sim$ on $T_r(H)$ by $x \sim y$ if $hx = hy$ for some (and thus for all) $h \in H$. Clearly, $\sim$ defines a congruence on $T_r(H)$;

denote the quotient $T_r(H)/\sim$ by $\Gamma_r(H)$. Hence, $\Gamma_r(H)$ is a monoid. Let $h_1, h_2 \in H, x, y \in T_r(H)$. Since $h_1x = h_1y$ implies that [x]=[y], it follows that for every $h_1, h_2 \in H$ there is a unique $[x] \in \Gamma_r(H)$ such that $h_1x = h_2$. Thus $[x] \rightarrow \rho_x|_H$ defines an isomorphism from $\Gamma_r(H)$ to $P(H)$, the regular group of permutations on H. Let H_1, H_2 be two $\mathcal{H}$-classes in the $\mathcal{D}$-class D. Then, by Lemma 1.5.1, there are two mutually inverse $\mathcal{R}$-class preserving inner translations between the two corresponding $\mathcal{L}$-classes, L_1 and L_2. Assume that these two mutual inverse inner translations are ρ_s, ρ_t. One can easily show that $\Gamma_r(H_1) \cong \Gamma_r(H_2)$ via the mapping $[x] \longmapsto [txs]$. Hence, $\Gamma_r(H)$ depends only on the $\mathcal{D}$-class D containing H and not on the chosen $\mathcal{H}$-class. The group $\Gamma_r(H)$ is called the *right Schützenberger group* of D. Dually, the *left Schützenberger group* $\Gamma_\ell(H)$ can be defined in a similar way. It follows from the fact that $\Gamma_r(H)$ and $\Gamma_\ell(H)$ are regular and the fact that they commute that $\Gamma_r(H) \cong \Gamma_\ell(H)$. Hence, we only consider $\Gamma_r(H)$ and call it the *Schützenberger* group. Note that if H is a group, then since each class $[x] \in \Gamma_r(H)$ has a unique representative x from H, it follows that $\Gamma_r(H) \cong H$. To summarize, we state the following theorem.

Theorem 6.1.1 *Let H_1 and H_2 be two $\mathcal{H}$-classes in a $\mathcal{D}$-class D. Then,*

1. $\Gamma_r(H_1) \cong \Gamma_r(H_2) \cong \Gamma_\ell(H_2) \cong \Gamma_\ell(H_1)$,
2. $|\Gamma_r(H)| = |H|$ *for each $\mathcal{H}$-class in D,*
3. *if D is regular, then $\Gamma(D)$ is isomorphic to each of its maximal subgroups.*

Further details on the Schützenberger group can be found in, for example, [30].

In this chapter, we study the Schützenberger group of non-regular $\mathcal{D}$-classes and show that, in some cases, the Schützenberger group of a non-regular $\mathcal{D}$-class is isomorphic to that of some regular $\mathcal{D}$-class.

6.2 The commutative finite semigroups

In this section we show that for each non-regular $\mathcal{D}$-class, in a commutative finite semigroup, there is always a regular $\mathcal{D}$-class such that the *Schützenberger* group of the non-regular one is a homomorphic image of that of the regular; and with certain condition they become isomorphic.

Let us first agree on some of the notation that we use in this section.

Notation 6.2.1

1. For x, y in a semigroup $\mathcal{S}$ we define $T(x,y)$ as follows:

$$T(x,y) = \{v \in \mathcal{S} : xv = y\}.$$

2. Let $\mathcal{S}$ be a semigroup, let H be an $\mathcal{H}$-class in $\mathcal{S}$, and let H^* be a regular $\mathcal{H}$-class in $\mathcal{S}$. Then, by $[x]$ and $[x]^*$ we denote the equivalence classes of x with respect to $\sim$ on $T_r(H)$ and on $T_r(H^*)$ respectively.

Our main goal is to prove the following theorem:

Theorem 6.2.1 *If $\mathcal{S}$ is a commutative finite semigroup, then for each non-singleton $\mathcal{H}$-class H in $\mathcal{S}$ there exists at least one regular $\mathcal{H}$-class H^* in $\mathcal{S}$ such that*

1. *$\Gamma_r(H)$ is a homomorphic image of $\Gamma_r(H^*) \cong H^*$,*
2. *the identity of H^* belongs to $T_r(H)$,*
3. *$\Gamma_r(H) \cong \Gamma_r(H^*)$ if and only if*

$$H^* \cap [u] = \{ u \}$$

where u is the identity of H^*.

In order to prove the above theorem we need the following lemmas.

Lemma 6.2.1 *Let x, y be elements of an $\mathcal{H}$-class H in a semigroup $\mathcal{S}$ and $xx_0 = y$ $(x_0x = y)$ for some $x_0 \in \mathcal{S}$. Then $x_0 \in T_r(H)$ $(x_0 \in T_\ell(H))$.*

Remark 6.2.1 Since $x \,\mathcal{H}\, y$, x_0 always exists.

Proof. We first show that $Hx_0 \subseteq H$. So let $z \in Hx_0$. Then $z = hx_0$ for some $h \in H$ and

$$S^1z = S^1hx_0 = S^1xx_0 = S^1y.$$

Hence,

$$z \in \mathcal{L}_y. \tag{6.1}$$

Since $h, x \in H$, there is a $t \in S^1$ such that $h = tx$ and therefore

$$zS^1 = hx_0S^1 = txx_0S^1 = tyS^1 = txS^1 = hS^1 = yS^1.$$

Hence,

$$z \in \mathcal{R}_y. \tag{6.2}$$

By (6.1) and (6.2) $z \in \mathcal{L}_y \cap \mathcal{R}_y = H$ and $Hx_0 \subseteq H$.

To show that $H \subseteq Hx_0$, let $h \in H$. Since $h, y \in H$ there is a $t \in S^1$ such that $h = ty = txx_0$. Thus

$$txS^1 = tyS^1 = hS^1 = yS^1$$

which shows that

$$tx \in \mathcal{R}_y. \tag{6.3}$$

Also since $x, y \in H$, there is a $q \in S^1$ such that $x = yq$. Thus

$$S^1tx = S^1tyq = S^1hq = S^1yq = S^1x = S^1y$$

which shows that

$$tx \in \mathcal{L}_y. \tag{6.4}$$

From (6.3) and (6.4) it follows that $tx \in \mathcal{L}_y \cap \mathcal{R}_y = H$ and $txx_0 \in Hx_0$. Hence, $h \in Hx_0$ and $H \subseteq Hx_0$. It then follows that $Hx_0 = H$. ∎

Corollary 6.2.1 *Let H be an $\mathcal{H}$-class in a semigroup $\mathcal{S}$ and $x \in \mathcal{S}$. Then,*

$$\text{either } H \cap Hx = \emptyset \text{ or } Hx = H \ (H \cap xH = \emptyset \text{ or } xH = H).$$

Corollary 6.2.2 *Let x, y be elements in an $\mathcal{H}$-class H. Then*

$$T(x,y) \subseteq T_r(H).$$

Lemma 6.2.2 *Let H be a finite $\mathcal{H}$-class in a semigroup $\mathcal{S}$, and let $u \in T_r(H)$. Then,*

$$\exists r \in \mathbf{N} : \ (hu^r = h, \forall h \in H).$$

Proof. Since $u \in T_r(H)$, it follows that

$$(\forall i \in \mathbf{N}) \ \ Hu^i = H \Longrightarrow (\forall i \in \mathbf{N}, \ \forall h \in H) \ \ hu^i \ \in H.$$

Since H is finite, there exist $i, j \in \mathbf{N}$ such that $hu^i = hu^j$. Without lost of generality we may assume that $j = i + r : r > 0$. It then follows that

$$\exists h' \in H : hu^i = hu^j = hu^{i+r} = h'$$

$$\Longrightarrow h' = hu^{i+r} = hu^{i+2r} = h'u^r$$

$$\Longrightarrow (\forall h \in H) \ hu^r = h. \qquad \blacksquare$$

The next lemma follows immediately from the proof of Lemma 6.2.2.

Lemma 6.2.3 *Let H be a finite $\mathcal{H}$-class in a semigroup $\mathcal{S}$, and let $u \in T_r(H)$. Then,*

$$(\forall h \in H, i \neq j) \ hu^i = hu^j \Longleftrightarrow hu^{|i-j|} = h.$$

Remark 6.2.2 Note that Lemmas 6.2.1, 6.2.2 and 6.2.3 don't depend on $\mathcal{S}$ being finite or commutative.

Notation 6.2.2 In the rest of this section, H stands for an $\mathcal{H}$-class of order m in a commutative finite semigroup $\mathcal{S}$. Also, $h_1, h_2, \ldots, h_m$ stand for the elements of H, i.e.

$$H = \{ h_1,\ h_2, \ldots,\ h_m \}.$$

Also, $u_i \in \mathcal{S}^1$, $r_i \in \mathbf{N}$ stand for the corresponding elements associated with h_i such that $h_i = h_1 u_i$ and $u_i^{r_i}$ is an idempotent. Finally, u stands for $u_1^{r_1} u_2^{r_2} \cdots u_n^{r_n}$ and H^* stands for the $\mathcal{H}$-class containing u.

Clearly, u is an idempotent. By Corollary 1.5.1 H^* is a subgroup of $\mathcal{S}$ and u is its identity.

Lemma 6.2.4 *For the $\mathcal{H}$-classes H and H^*, we have*

$$HH^* = H = H^*H.$$

Proof. By Lemma 6.2.2 $h = hu_i^{r_i}$ for all $h \in H$ and all $i : 1 \leq i \leq n$. Thus $H = Hu \subseteq HH^*$.

To show that $HH^* \subseteq H$, let $z \in HH^*$ so that $z = hh^*$ for some $h \in H$, $h^* \in H^*$ and therefore

$$\mathcal{S}^1 z = \mathcal{S}^1 hh^* = \mathcal{S}^1 h^* h = \mathcal{S}^1 uh = \mathcal{S}^1 hu = \mathcal{S}^1 h = \mathcal{S}^1 h_1 \Longrightarrow z \in \mathcal{L}_{h_1}. \quad (6.5)$$

Since $\mathcal{S}$ is commutative, it follows from (6.5) that $z \in \mathcal{L}_{h_1} \cap \mathcal{R}_{h_1} = H$. Hence,

$$HH^* = H = H^*H. \quad \blacksquare$$

Lemma 6.2.5 *The right stabilizer of H contains the right stabilizer of H^*, that is*

$$T_r(H^*) \subseteq T_r(H).$$

Proof. Let $v \in T_r(H^*)$. Then $H^*v = H^*$. By Lemma 6.2.3 $HH^* = H$. Thus,

$$Hv = HH^*v = HH^* = H \Longrightarrow v \in T_r(H).$$

Hence, $T_r(H^*) \subseteq T_r(H)$. $\blacksquare$

Lemma 6.2.6 *The right stabilizer of H^* contains H^*, i.e.*

$$H^* \subseteq T_r(H^*).$$

Proof. The result of this lemma follows immediately from the fact that H^* is a group. ■

Lemma 6.2.7 *For any $x \in T_r(H)$ there exists $y \in T_r(H^*)$ such that $[x] = [y]$.*

Proof. Let $x \in T_r(H)$. Then $Hx = H$ and

$$h_1 x = h_j = h_1 u_j = h_1 u u_j = h_1 u_1^{r_1} u_2^{r_2} \cdots u_{j-1}^{r_{j-1}} u_j^{r_j+1} u_{j+1}^{r_{j+1}} \cdots u_n^{r_n}. \tag{6.6}$$

Let $y = u_1^{r_1} u_2^{r_2} \cdots u_{j-1}^{r_{j-1}} u_j^{r_j+1} u_{j+1}^{r_{j+1}} \cdots u_n^{r_n}$. Then

$$\mathcal{S}^1 u_j \subseteq \mathcal{S}^1 \Longrightarrow \mathcal{S}^1 u_j u \subseteq \mathcal{S}^1 u \Longrightarrow \mathcal{S}^1 y \subseteq \mathcal{S}^1 u, \tag{6.7}$$

and

$$\begin{aligned} \mathcal{S}^1 u_1^{r_1} \cdots u_{j-1}^{r_{j-1}} u_j^{r_j-1} u_{j+1}^{r_{j+1}} \cdots u_n^{r_n} \subseteq \mathcal{S}^1 &\Longrightarrow \mathcal{S}^1 u \subseteq \mathcal{S}^1 u_j \\ &\Longrightarrow \mathcal{S}^1 u^2 \subseteq \mathcal{S}^1 u_j u \Longrightarrow \mathcal{S}^1 u \subseteq \mathcal{S}^1 y. \end{aligned} \tag{6.8}$$

From (6.7) and (6.8) it follows that $\mathcal{S}^1 u = \mathcal{S}^1 y$ and since $\mathcal{S}$ is commutative, $u\mathcal{S}^1 = y\mathcal{S}^1$ and thus $y \in H^*$. By Lemmas 6.2.5 and 6.2.6 $H^* \subseteq T_r(H^*) \subseteq T_r(H)$. It then follows from (6.6) that $[x] = [y]$. ■

Remark 6.2.3 Lemma 6.2.7 says that

$$\Gamma_r(H) \;=\; \{ \;\; [x] \;:\; x \in T_r(H^*) \; \}.$$

Lemma 6.2.8 *Let $z \in T_r(H^*)$. Then,*

$$[z]^* \subseteq \; [z].$$

Proof. Let $w \in [z]^*$. Then $hz = hw$ for some $h \in H^*$. Let $v \in H$ be chosen arbitrarily. Then, $vhz = vhw$. Since $HH^* = H = H^*H$ (Lemma 6.2.4), $vh = h_i$ for some $h_i \in H$. Thus $h_i z = h_i w$. Since $z, w \in T_r(H^*) \subseteq T_r(H)$ and $h_i \in H$, it follows that $z \sim w$ in $T_r(H)$. Hence, $w \in [z]$ and $[z]^* \subseteq [z]$. ∎

The next lemma is an immediate consequence of the fact that H^* and $\Gamma_r(H^*)$ are isomorphic.

Lemma 6.2.9 $\Gamma_r(H^*) = \{ [x]^* : x \in H^* \}$.

Proof of Theorem 6.2.1 Define $\alpha : \Gamma_r(H^*) \longrightarrow \Gamma_r(H)$ by

$$([x]^*) \alpha = [x].$$

Since $[x]^* \subseteq [x]$, α is a well-defined homomorphism. To prove that α is onto, suppose that $[y] \in \Gamma_r(H)$. Then by Lemma 6.2.7 (Remark 6.2.3) there exists $z \in T_r(H^*)$ such that $[z] = [y]$. Thus $z \in T_r(H^*)$ and $[z]^*$ is in $\Gamma_r(H^*)$ and

$$([z]^*) \alpha = [z] = [y].$$

This establishes that α is surjective and $\Gamma_r(H)$ is a homomorphic image of $\Gamma_r(H^*)$.

(2) Follows from Lemmas 6.2.5 and 6.2.6.

(3) Assume that $\Gamma_r(H)$ and $\Gamma_r(H^*)$ are isomorphic and $x \in H^* \cap [u]$. Since α is surjective and $\Gamma_r(H)$ is isomorphic to $\Gamma_r(H^*)$, α is an isomorphism and thus,

$$([x]^*)\alpha = [x] = [u] = ([u]^*)\alpha.$$

Since α is injective, we must have $[x]^* = [u]^*$. It then follows that $h^*x = h^*u$ for some $h^* \in H^*$. Since $x, u \in H^*$ and u is the identity of the maximal subgroup H^* , it follows that $x = u$ and thus $H^* \cap [u] = \{u\}$.

Conversely, if $H^* \cap [u] = \{u\}$ and $([h_1]^*)\alpha = ([h_2]^*)\alpha$ for some $h_1, h_2 \in H^*$ then

$$[h_1] = [h_2] \Longrightarrow gh_1 = gh_2 \text{ for all } g \in H \Longrightarrow gh_1h_1^{-1} = gh_2h_1^{-1}$$

$$\Longrightarrow gu = gh_2h_1^{-1} \text{ (as } u \text{ is the identity)} \Longrightarrow h_2h_1^{-1} \in [u]$$

$$\Longrightarrow h_2h_1^{-1} \in H^* \cap [u] = \{u\} \Longrightarrow u = h_2h_1^{-1} \Longrightarrow h_1 = h_2.$$

Hence α is an isomorphism. ■

Theorem 6.2.2 *If $\mathcal{S}$ is a commutative finite semigroup, then the following are equivalent:*

1. *all the $\mathcal{H}$-classes of $\mathcal{S}$ are regular,*
2. *for each generator g of $\mathcal{S}$ there exists an integer i such that $g^{i+1} = g$,*
3. *for each $x \in \mathcal{S}$ there exists $j \in \mathbf{N} : x^{j+1} = x$.*

Proof. Suppose that 1 holds and that x is a generator of $\mathcal{S}$. Then H_x is a group. Since $\mathcal{S}$ is finite, $x^i = e$ for some i in $\mathbf{N}$. Hence, $x^{i+1} = x$ and 2 is established.

Assume that 2 holds and $x = g_1g_2 \cdots g_n$. Since $\mathcal{S}$ is commutative, $x^{i+1} = x$ where $i = \text{l.c.m.}(r_1, r_2, \ldots r_n) : g_i^{r_i+1} = g_i$. Thus, 2 implies 3.

To show that 3 implies 1, assume that 3 holds, $x \in \mathcal{S}$ and $x^{r+1} = x$. Then x^r is an idempotent and H_{x^r} is a maximal subgroup. Also, $\{\, x,\, x^2, \ldots, x^r \,\}$ is a subgroup of H_{x^r}. Hence, $H_x = H_{x^r}$. ■

Theorem 6.2.3 *Let H^* and H be finite $\mathcal{H}$-classes in a commutative semigroup $\mathcal{S}$ such that H^* is regular, the identity, e, of H^* belongs to $T_r(H)$ and $\Gamma_r(H)$ is a homomorphic image of H^*. If $[e] \cap H^* = \{e\}$, then $H^* \cong \Gamma_r(H)$.*

Proof. Assume that $[e] \cap H^* = \{e\}$. Fix $h_1 \in H$ and let $h_1^*, h_2^* \in H^*$. Then, $h_1 h_1^* = h_1 h_2^*$ implies that

$$h_1 e = h_1 h_2^* (h_1^*)^{-1}.$$

Hence, $h_2^*(h_1^*)^{-1} \in [e]$ and $h_2^*(h_1^*)^{-1} = e$ which implies that $h_1^* = h_2^*$. This means that $|h_1 H^*| = |H^*|$ and since $h_1 H^* \subseteq H$, we get $|\Gamma_r(H)| \geq |H^*|$. Since $\Gamma_r(H)$ is a homomorphic image of H^*, we deduce that $|H^*| = |H| = |\Gamma_r(H)|$ and thus $H^* \cong \Gamma_r(H)$. ∎

Theorem 6.2.1 can be modified for infinite commutative semigroups as follows:

Theorem 6.2.4 *Let H be a non-singleton finite $\mathcal{H}$-class in a commutative semigroup $\mathcal{S}$. If H contains an element h_1 such that*

$$\forall\ h \in H\ (\exists v \in \mathcal{S}^1, r \in \mathbf{N} : h_1 v = h, v^r \in E(\mathcal{S})),$$

then there exists a regular $\mathcal{H}$-class, H^* such that

1. *$\Gamma_r(H)$ is a homomorphic image of $\Gamma_r(H^*) \cong H^*$,*
2. *the identity of H^* belongs to $T_r(H)$,*
3. *$\Gamma_r(H) \cong \Gamma_r(H^*)$ if and only if*

   $$H^* \ \cap\ [u] \ =\ \{\ u\ \}$$

   where u is the identity of H^*.

We end this section with the following example.

Example 6.2.1 Let $\mathcal{S}$ be the semigroup defined by the presentation

$$\langle\ a, b : a^4 = 1, b^2 = 0, ab = ba, a^2 b = b\ \rangle.$$

Then $\mathcal{S}$ is a commutative semigroup with seven elements:

$$\{ 0, 1, a, a^2, a^3, b, ab \},$$

and three $\mathcal{D}$-classes ($\mathcal{H}$-classes) :

$$H_1 = \{ 1, a, a^2, a^3 \}, \ H_2 = \{ b, ba \}, \ H_3 = \{ 0 \}.$$

Also, $\mathcal{S}$ has two idempotents 1 and 0, so that H_1 and H_3 are regular and H_2 is not. From $|H_2| = 2$ it follows that

$$\Gamma_r(H_2) \cong C_2$$

the cyclic group of order 2, while the other two Schützenberger groups are clearly the trivial group and C_4. Hence,

$$\Gamma_r(H_1) \ncong \Gamma_r(H_2) \ncong \Gamma_r(H_3).$$

Since $\mathcal{S}$ is commutative, Theorem 6.2.1 implies that $\Gamma_r(H_2)$ is a homomorphic image of $\Gamma_r(H_1)$. Also,

$$T_r(H_2) = \{1, a, a^2, a^3\}.$$

Hence,

$$\Gamma_r(H_2) = \{ \{a, a^3\}, \{1, a^2\}\}$$

and $[1] \cap H_1 = \{1, a^3\} \neq \{1\}$ in accordance with Part (3) of Theorem 6.2.1.

6.3 Finite semigroups with $xyx = x^2y$ for all x, y or $xyx = yx^2$ for all x, y

In this section, we consider the Schützenberger groups of the $\mathcal{D}$-classes in a finite semigroup, $\mathcal{S}$, satisfying $xyx = yx^2$ for all $x, y \in \mathcal{S}$. We show that Theorem 6.2.1 holds in these types of semigroups and in their duals which satisfy $xyx = yx^2$.

Remark 6.3.1 Other aspects of these types of semigroups are considered in Chapter 7.

Theorem 6.3.1 *If $\mathcal{S}$ is a finite semigroup such that $xyx = x^2y$ for all $x, y \in \mathcal{S}$, then for each non-singleton $\mathcal{H}$-class H in $\mathcal{S}$ there exists at least one regular $\mathcal{H}$-class H^* such that*

1. *$\Gamma_r(H)$ is a homomorphic image of $\Gamma_r(H^*) \cong H^*$,*
2. *the identity of H^* belongs to $T_r(H)$,*
3. *$\Gamma_r(H) \cong \Gamma_r(H^*)$ if and only if*

$$H^* \cap [u] = \{ u \}$$

where u is the identity of H^*.

Proof. Assume that

$$H = \{h_1, h_2, \ldots, h_n\}.$$

As in Notation 6.2.2, let $u_i \in \mathcal{S}^1, r_i \in \mathbf{N}$ such that $h_i = u_i h_1$ and $u_i^{r_i}$ is an idempotent for all $i : 1 \leq i \leq n$, let $u = u_1^{r_1} u_2^{r_2} \cdots u_n^{r_n}$, and let H^* be the $\mathcal{H}$-class containing u. Also, $[x]^*$ and $[x]$ stand for the equivalence classes of x with respect to $T_r(H^*)$ and $T_r(H)$ respectively. Then, clearly,

$$(\forall i : 1 \leq i \leq n) \; uh_i = h_i.$$

Therefore,

$$(\forall i : 1 \leq i \leq n) \; h_i u = uh_i u = u^2 h_i = uh_i = h_i.$$

Hence, $H \subseteq HH^*$.

Also, if $x \in HH^*$, then $x = hh^*$ for some $h \in H$ and $h^* \in H^*$. Hence,

$$\mathcal{S}^1 hh^* = \mathcal{S}^1 uhh^* = \mathcal{S}^1 h^* hh^* = \mathcal{S}^1 h^* h^* h = \mathcal{S}^1 uh = \mathcal{S}^1 h,$$

and

$$hh^*\mathcal{S}^1 = hu\mathcal{S}^1 = h\mathcal{S}^1.$$

Thus, $hh^* \in H$ and $HH^* = H$. It then follows that

$$T_r(H^*) \subseteq T_r(H). \tag{6.9}$$

Furthermore, for each $x \in T_r(H)$ there exists $y \in T_r(H^*)$ such that $[x] = [y]$ because if $x \in T_r(H)$, then

$$h_1 x = h_j = u_j h_1 = u_j(h_1 u) = u_j h_1 u_1^{r_1} \cdots u_{j-1}^{r_{j-1}} u_j^{2r_j} u_{j+1}^{r_{j+1}} \cdots u_n^{r_n}$$

$$= u_j^{r_j} h_1 u_1^{r_1} u_2^{r_2} \cdots u_{j-1}^{r_{j-1}} u_j^{r_j+1} u_{j+1}^{r_{j+1}} \cdots u_n^{r_n} = h_1 u_1^{r_1} u_2^{r_2} \cdots u_{j-1}^{r_{j-1}} u_j^{r_j+1} u_{j+1}^{r_{j+1}} \cdots u_n^{r_n}.$$

and since the group generated by $u_1^{r_1} u_2^{r_2} \cdots u_{j-1}^{r_{j-1}} u_j^{r_j+1} u_{j+1}^{r_{j+1}} \cdots u_n^{r_n}$ contains u as its identity, $u_1^{r_1} \cdots u_{j-1}^{r_{j-1}} u_j^{r_j+1} u_{j+1}^{r_{j+1}} \cdots u_n^{r_n}$ must be in H^*. Hence, $[x] = [y]$ where $y = u_1^{r_1} \cdots u_{j-1}^{r_{j-1}} u_j^{r_j+1} u_{j+1}^{r_{j+1}} \cdots u_n^{r_n} \in H^* \subseteq T_r(H^*)$. This shows that

$$\Gamma_r(H) \ = \ \{\ [x] : x \in H^*\ \}. \tag{6.10}$$

Also, if $x \in T_r(H^*)$ and $z \in [x]^*$, then

$$(\forall\ h^* \in H^*)\ h^*x = h^*z$$

which implies that

$$(\forall\ i : 1 \leq i \leq n)\ h_i h^* x = h_i h^* z;$$

it then follows from the fact that $HH^* = H$ that

$$(\forall\ i : 1 \leq i \leq n)\ h_i x = h_i z.$$

This establishes that

$$(\forall\ x \in T_r(H^*)) \quad [x]^* \ \subseteq \ [x]. \tag{6.11}$$

Now, define $\alpha : H^* \longrightarrow \Gamma_r(H)$ by

$$(h^*)\alpha = [h^*].$$

It then follows from (6.10) and (6.11) that α is a well-defined onto homomorphism.

(2) Follows from the fact that $HH^* = H^*$.

(3) As in the proof of Theorem 6.2.1 (3). ∎

Similarly, one can show the following dual result of Theorem 6.3.1

Theorem 6.3.2 *If $\mathcal{S}$ is a finite semigroup such that $xyx = yx^2$ for all $x, y \in \mathcal{S}$, then for each non-singleton $\mathcal{H}$-class H in $\mathcal{S}$ there exists at least one regular $\mathcal{H}$-class H^* such that*

1. *$\Gamma_\ell(H)$ is a homomorphic image of $\Gamma_\ell(H^*) \cong H^*$,*
2. *the identity of H^* belongs to $T_\ell(H)$,*
3. *$\Gamma_\ell(H) \cong \Gamma_\ell(H^*)$ if and only if*

$$H^* \cap [u] = \{ u \}$$

where u is the identity of H^*.

Theorem 6.3.1 and 6.3.2 can be modified for finite $\mathcal{H}$-classes of infinite semigroups as in Theorem 6.2.4. Also, Theorem 6.2.3 holds in these types of semigroups.

Example 6.3.1 Consider the semigroup $\mathcal{S}$ defined by the following presentation:

$$P = \langle a, b \mid a^3 = a^2, b^3 = b, aba = a^2b, bab = b^2a \rangle.$$

Then $\mathcal{S}$ is of order 12 and

$$SG(P) = \{ a^i, b^j, a^ib^j, b^ja^i : 1 \leq i, j \leq 2 \}.$$

The set of idempotents of $\mathcal{S}$ is

$$E(\mathcal{S}) = \{\ a^2,\ b^2,\ a^2b^2,\ b^2a^2\ \}.$$

The $\mathcal{H}$-classes of $\mathcal{S}$ are:

$$H_1 = \{\ a\ \},\quad H_2 = \{\ a^2\ \},\quad H_3 = \{\ b,\ b^2\ \},\quad H_4 = \{\ ba,\ b^2a\ \},$$

$$H_5 = \{\ ab\ \}, H_6 = \{\ ab^2\ \},\ H_7 = \{\ a^2b,\ a^2b^2\ \},\ H_8 = \{\ ba^2,\ b^2a^2\ \}.$$

Clearly, H_4 is the only non-singleton non-regular $\mathcal{H}$-class and

$$T_r(H_4) = \{\ 1,\ b,\ b^2\},\quad \Gamma_r(H_4) = \{\{b\},\ \{1,b^2\}\} \cong H_3. \qquad \blacksquare$$

We end this chapter with a program to compute the Schützenberger group of a given $\mathcal{H}$-class.

The following program computes the Schützenberger group of a given $\mathcal{H}$-class, H. In order to show that there is a counter-example to Theorem 6.2.3 in general, the program is set up to compute the Schützenberger group of the H_{abc} in the semigroup $\mathcal{S}$ defined by

$$P = \langle\ a,b,c \mid a^3 = a,\ b^3 = b,\ c^3 = c,\ a^2ba = ab,\ a^2ca = ac,\ bc = cbc^2\ \rangle.$$

It is also set up to print the order of $\mathcal{S}$, the $\mathcal{H}$-class H_{abc}, $T_r(H_{abc})$, the set of idempotents in $T_r(H_{abc})$ and the regular $\mathcal{H}$-class $H_{a^2c^2}$. The number 16 is the representative of abc in the enumeration process. It can be executed only on the interface of Semi and Gap mentioned in 1.6.

```
a:=AbstractGenerator("a");;
b:=AbstractGenerator("b");;
c:=AbstractGenerator("c");;
I := [[a^3, a], [b^3, b], [c^3, c], [a^2 * b * a, a * b], [a^2 * c * a, a * c], [b * c, c * b * c^2]];
S:=SGEnumerate(I);;
idemp:=ListBlist([1..S.size],SGIdempotents(S));
```

```
H:=SGHClass(S,16); # the non regularℋ-class to be investigated,
TrH:=[ ];; # the right stabilizer of H,
for x in [1..S.size] do
        for y in H do
           z:=SGAct(S,y,x);
           if Intersection(H,[z]) <>[ ] then
              TrH:=Union(TrH,[x]); # by Corollary 6.2.1
           fi;
        od;
od;;
class:=[];
for x in TrH do
        class:=[x];
        h:=RandomList(H);
        for y in TrH do
           if SGAct(S,h,y)=SGAct(S,h,x) then
              class:=Union(class,[y]);
           fi;
        od;
        class:=[class];
        SchützenbergerGroup:=Union(SchützenbergerGroup,class);
od;;
Print("\n"The order of S",S.size,"\n"); Print("HClass16=",H,"\n");
Print("TrH=", TrH, "\n");
Print("The set of idempotents in TrH=", Intersection(idemp,TrH), "\n");
```

```
Print("SchützenbergerGroup=", SchützenbergerGroup, "\n");
Print("HClass37", SGHClass(S,37), "\n");
```

The output of the above described program is :

```
gap> The order of S =902
gap> HClass16=[ 16, 38 ]
gap> TrH=[ 1, 4, 15, 37 ]
gap> The set of idempotents in TrH is [ 4, 37 ]
gap> SchutzenbergerGroup=[ [ 1, 15 ], [ 4, 37 ] ]
gap> HClass37=[ 9, 10, 15, 37 ]
```

Enumerating $\mathcal{S}$ by Semi to find the corresponding words of the above output, we get:

$$H_{abc} = \{ \ abc, \ a^2cb \ \}, \quad T_r(H_{abc}) = \{ \ a, \ a^2, \ ac^2, \ a^2c^2 \ \},$$

$$\Gamma_r(H_{abc}) = \{ \ \{a, \ ac^2\}, \ \{a^2, \ a^2b^2\}\},$$

$$H_{a^2c^2} = \{ \ a^2c, \ ac, \ ac^2, \ a^2c^2 \ \},$$

and the idempotents in $T_r(H_{abc})$ are a^2 and a^2c^2.

Clearly, the idempotent a^2b^2 belongs to $T_r(H_{abc})$ and the identity of $\Gamma_r(H_{abc})$ is $\{a^2, \ a^2b^2\}$ which intersects $H_{a^2b^2}$ only in a^2b^2. Define $\alpha : H_{a^2b^2} \longrightarrow \Gamma_r(H_{abc})$ by:

$$(x)\alpha = \begin{cases} \{a, ac^2\} : x \in \{ac, a^2c\}, \\ \{a^2, a^2c^2\} \quad \text{otherwise.} \end{cases}$$

It then follows by Lemma 4.1.1 that α is a homomorphism. Hence, $\Gamma_r(H_{abc})$ is a homomorphic image of $H_{a^2b^2}$ but $\Gamma_r(H_{abc}) \ncong H_{a^2b^2}$.

Chapter 7

Semi-commutative semigroups

7.1 Introduction

To generalize some of the results of commutative semigroups to a larger class, J. L. Chrislock [14] studied what was called *medial semigroups* (see Definition 7.1.1 below). Then T. Tamura and J. Shafer [50] generalized some results of commutative semigroups to *exponential semigroups* (see Definition 7.1.2 below). For the same purpose, T. Nordahl [36] studied what was called *E-m semigroups* (see Definition 7.1.3). Followed by H. Lal [29] and C. Nagore [34], N. Mukherjee [33] introduced and studied quasi commutative semigroups (see Definition 7.1.4). *σ-reflexive semigroups,* (see Definition 7.1.5), were defined and studied by M. Chacron and G. Thierrin [12] and later by A. Cherubini and A. Varisco [13]. *Weakly commutative semigroups,* (see Definition 7.1.6) were studied by M. Petrich [38], A. Cherubini and A. Varisco [13], N. Kehaypulu [27] and by N. Kehaypulu, P. Kiriakuli, S. Hanumantha and P. Lakshmi [28].

Definition 7.1.1 A semigroup $\mathcal{S}$ is *medial* if $xaby = xbay$ for all $x, a, b, y \in \mathcal{S}$.

Definition 7.1.2 A semigroup $\mathcal{S}$ is *exponential if* $(xy)^n = x^n y^n$ *for all positive integers* n *and all* x, y *in* $\mathcal{S}$.

Definition 7.1.3 A semigroup $\mathcal{S}$ is *E-m* if $(xy)^m = x^m y^m$ for all $x, y \in \mathcal{S}$.

Hence, *medial semigroups* are E-2.

Definition 7.1.4 A semigroup $\mathcal{S}$ is quasi commutative if for any two elements x, y in $\mathcal{S}$, $xy = y^r x$ for some positive integer r.

Definition 7.1.5 A subsemigroup T of a semigroup $\mathcal{S}$ is called reflexive if, for every $a, b \in \mathcal{S}$, $ab \in T$ implies $ba \in T$. A semigroup $\mathcal{S}$ is σ-reflexive if all the subsemigroups of $\mathcal{S}$ are reflexive.

Definition 7.1.6 *A semigroup* $\mathcal{S}$ *is called weakly commutative* if, for every $a, b \in \mathcal{S}$, $(ab)^n = bxa$ for some $x \in \mathcal{S}$ and $n \in \mathbf{N}$.

The next proposition gives an equivalent condition of *σ-reflexive semigroups.*

Proposition 7.1.1 ([12]). *A semigroup $\mathcal{S}$ is σ-reflexive if and only if,*

$$(\forall\ a, b \in \mathcal{S},\ \exists m \in \mathbf{N}) : ab = (ba)^m.$$

Motivated by our attempt to extend Theorem 6.2.1 to a larger class than the class of commutative semigroups, we introduced the semigroups satisfying $xyx = x^2y$ and those satisfying $xyx = yx^2$ in Chapter 6. In this chapter, we study these types of semigroups in more detail. First, note that these semigroups are in the families of E-2 and *exponential semigroups.* We call the first type R-semicommutative and the second type L-semi-commutative semigroups. We will see that each $\mathcal{R}$-class of an R-semi-commutative semigroup is commutative and each $\mathcal{L}$-class of an L-semi-commutative semigroup is commutative. The main goal of this chapter is to show that there are many properties shared by these classes and commutative semigroups; the results of this chapter are examples of some of these properties.

Definition 7.1.7 A subset X of a semigroup $\mathcal{S}$ is called *L-semi-commutative* if and only if $xyx = yx^2$ for every x, y in X. Dually, X is called *R-semi-commutative* if and only if $xyx = x^2y$ for every x, y in X. X is called *semi-commutative* if and only if it is both R and L-semi-commutative.

Recall that an element x in a semigroup $\mathcal{S}$ is said to be of *index* $r \in \mathbf{N}$ if there exists $k \in \mathbf{N}$ such that $x^{k+r} = x^r$. A subset X of $\mathcal{S}$ is said to be of *index* r if each of its elements is of index r.

The next proposition proves that if $\mathcal{S}$ is a semigroup generated by an L-semi-commutative (R-semi-commutative or both) set of index 1, then $\mathcal{S}$ is L-semi-commutative (R-semi-commutative or both).

Proposition 7.1.2 *Let $\mathcal{A}$ be a generating set of a semigroup $\mathcal{S}$ such that $\mathcal{A}$ is of index 1. Then $\mathcal{S}$ is L-semi-commutative if and only if $\mathcal{A}$ is L-semi-commutative.*

Proof. We first show that $axa = xa^2$ for every $x \in \mathcal{S}, a \in \mathcal{A}$. So let $x \in \mathcal{S}$, $a \in \mathcal{A}$. Then $x = a_1a_2 \cdots a_q$ where $a_i \in \mathcal{A}$ and $q \in \mathbf{N}$. Assume that $\mathcal{A}$ is L-semi-commutative. Since each generator is of index 1, there exists an integer $m \in \mathbf{N}$ such that $a^{m+1} = a$. We use mathematical induction on q. When $q = 1$, according to our hypothesis,

$$axa = aa_1a = a_1a^2 = xa^2.$$

Assume that $axa = xa^2$ holds for every integer $q : 1 \leq q \leq k \in \mathbf{N}$. We must now verify that $axa = xa^2$ holds with k replaced by $k+1$, assuming it holds for k. Hence, we have

$$axa = aa_1a_2 \cdots a_ka_{k+1}a = aa_1a_2 \cdots a_ka^ma_{k+1}a$$

$$= a_1a_2\cdots a_kaa_{k+1}a = a_1a_2\cdots a_ka_{k+1}a^2.$$

Hence, $axa = xa^2$ for every $x \in \mathcal{S},\ a \in \mathcal{A}$.

Let $x, y \in \mathcal{S}$. Then $x = a_1a_2\cdots a_p,\ y = b_1b_2\cdots b_q$ where $a_i, b_i \in \mathcal{A}$; and

$$\begin{aligned} xyx &= a_1a_2\cdots a_p(b_1b_2\cdots b_q)a_1a_2\cdots a_p \\ &= a_1\underbrace{(a_2\cdots a_pb_1b_2\cdots b_q)}_{z\in\mathcal{S}}a_1a_2\cdots a_p \\ &= a_2(a_3\cdots a_pb_1b_2\cdots b_qa_1^2)a_2\cdots a_p \\ &= a_3\cdots a_pb_1b_2\cdots b_qa_1^2a_2^2a_3\cdots a_p \\ &= \cdots = ya_1^2a_2^2\cdots a_p^2 = yx^2. \end{aligned}$$

This establishes that $\mathcal{S}$ is L-semi-commutative. ■

This result has the following dual:

Proposition 7.1.3 *Let $\mathcal{A}$ be a generating set of a semigroup $\mathcal{S}$ such that $\mathcal{A}$ is of index 1. Then $\mathcal{S}$ is R-semi-commutative if and only if $\mathcal{A}$ is R-semi-commutative.*

We no immediately deduce

Corollary 7.1.1 *Let $\mathcal{A}$ be a generating set of a semigroup $\mathcal{S}$ such that $\mathcal{A}$ is of index 1. Then $\mathcal{S}$ is semi-commutative if and only if $\mathcal{A}$ is semi-commutative.*

The next lemma shows that the $\mathcal{R}$-equivalent divides the R-semi-commutative semigroup $\mathcal{S}$ into blocks of commutative R-classes.

Lemma 7.1.1 *Let $\mathcal{S}$ be an R-semi-commutative semigroup. If R is an $\mathcal{R}$-class in $\mathcal{S}$, then R is commutative.*

Proof. Suppose that $x, y \in R$. Then there exists $t \in \mathcal{S}^1$ such that $x = yt$ and thus

$$xy = yty = y^2t = y(yt) = yx.$$

This shows that R is commutative. ∎

Lemma 7.1.2 *Let $\mathcal{S}$ be an L-semi-commutative semigroup. If L is an $\mathcal{L}$-class in $\mathcal{S}$, then L is commutative.*

Proof. Suppose that $x, y \in L$. Then there exists $t \in \mathcal{S}^1$ such that $y = tx$ and

$$xy = xtx = tx^2 = (tx)x = yx.$$

Hence, L is commutative. ∎

We now immediately deduce

Corollary 7.1.2 *If $\mathcal{S}$ is semi-commutative, then the $\mathcal{R}$-classes, $\mathcal{L}$-classes and $\mathcal{H}$-classes are commutative.*

In the next lemma we show that if every generator of $\mathcal{S}$ is of finite period then $\mathcal{S}$ is finite.

Lemma 7.1.3 *Let $\mathcal{A}$ be a generating set of a finitely generated R-semi-commutative semigroup $\mathcal{S}$ such that for each $x \in \mathcal{A}$ there exist two integers $r_x, t_x \in \mathbf{N}$ satisfying $x^{r_x+t_x} = x^{t_x}$. Then $\mathcal{S}$ is finite.*

Proof. Let $x \in \mathcal{S}$. Then $x = a_1a_2\cdots a_n : a_i \in \mathcal{A}$ and

$$x^i = a_1^i a_2^i \cdots a_n^i.$$

Since every $a \in \mathcal{A}$ is of finite period, there exist two integers $r, t \in \mathbf{N}$ such that $x^{r+t} = x^t$. This means that every $x \in \mathcal{S}$ is of finite period. Since $\mathcal{S}$ is finitely generated, it follows that $\mathcal{S}$ is finite. ∎

The next lemmas shows that in a R-semi-commutative (L-semi-commutative) semigroup each $\mathcal{R}$-class ($\mathcal{L}$-class) contains at most one idempotent.

Lemma 7.1.4 *Each $\mathcal{R}$-class in an R-semi-commutative semigroup $\mathcal{S}$ contains at most one idempotent.*

Proof. Suppose that $e, f \in E(\mathcal{S})$ and $e \,\mathcal{R}\, f$. Then e and f are left identities for the $\mathcal{R}$-class that contains e and f (Proposition 1.6.1). Hence,

$$ef = f; \; fe = e.$$

By Lemma 7.1.1 the $\mathcal{R}$-class containing e and f is commutative. Thus,

$$e = fe = ef = f.$$

This establishes the result of this lemma. ■

The proof of the next lemma is similar to the proof of Lemma 7.1.4.

Lemma 7.1.5 *Each $\mathcal{L}$-class in an L-semi-commutative semigroup $\mathcal{S}$ contains at most one idempotent.*

Next we show that the set of idempotents in an (L) R-semi-commutative semigroup is closed.

Recall that a semigroup, in which every element is an idempotent, is called a *band* and a commutative semigroup, in which every element is an idempotent, is called a *semilattice.*

Lemma 7.1.6 *If $\mathcal{S}$ is an R or L-semi-commutative semigroup, then $E(\mathcal{S})$ is a band.*

Proof. It is enough to show that $E(\mathcal{S})$ is closed. So let $e, f \in E(\mathcal{S})$. Then,

$$(ef)^2 = efef = e^2 f^2 = ef;$$

Hence, $E(\mathcal{S})$ is closed. ■

Definition 7.1.8 An element u of a monoid M is a *unit* when $uv = vu = 1$ holds for some $v \in M$.

The next lemma shows that the set of units in an R or L-semi-commutative monoid is commutative.

Lemma 7.1.7 *If $\mathcal{S}$ is an R or L-semi-commutative monoid, then the set of units in $\mathcal{S}$ is commutative.*

Proof. Let x, y be two units in $\mathcal{S}$. Then there exist $x', y' \in \mathcal{S}$ such that

$$xx' = x'x = 1 = yy' = y'y.$$

Suppose that $\mathcal{S}$ is R-semi-commutative. Then,

$$xy = 1xy = (yy')xy = y^2y'x = y(1)x = yx.$$

Similarly, if $\mathcal{S}$ is L-semi-commutative then

$$xy = xy(1) = xy(xx') = yx^2x' = yx(1) = yx.$$

This establishes the result of this lemma. ∎

It is easy to show that the congruence ρ defined on a commutative semigroup $\mathcal{S}$ by

$$x \ \rho \ y \Longleftrightarrow ax = ay \ \text{ for some } a \in \mathcal{S}$$

is the smallest cancellative congruence on $\mathcal{S}$. The next proposition shows that if $\mathcal{S}$ is R-semi-commutative then ρ is also the smallest cancellative congruence on $\mathcal{S}$.

Proposition 7.1.4 *Let $\mathcal{S}$ be an R-semi-commutative semigroup. Then the relation ρ defined on $\mathcal{S}$ by*

$$x \ \rho \ y \Longleftrightarrow ax = ay \ \textit{ for some } a \in \mathcal{S}$$

is the smallest cancellative congruence on $\mathcal{S}$.

Proof. Clearly, ρ is symmetric and reflexive. To show that ρ is transitive, let $(x,y),(y,z) \in \rho$. Then, there exist $a, b \in \mathcal{S}$ such that $ax = ay$ and $by = bz$. Hence,

$$(aby)(ax)(by) = (aby)(ay)(bz) \Longrightarrow a^2b^2y^2x = a^2b^2y^2z.$$

This shows that ρ is transitive. Now, let $(x,y) \in \rho,\ c, d \in \mathcal{S}$. Then $ax = ay$ for some $a \in \mathcal{S}$ and

$$ac(ax)d = ac(ay)d \Longrightarrow a^2(cxd) = a^2(cyd).$$

Hence, ρ is a congruence on $\mathcal{S}$. To show that ρ is cancellative, assume that $(wxz, wyz) \in \rho$. Then, $awxz = awyz$ for some $a \in \mathcal{S}$ and

$$(z^2aw)x = z(awxz) = z(awyz) = (z^2aw)y \Longrightarrow x\ \rho\ y.$$

This shows that ρ is cancellative. If ρ' is a cancellative congruence on $\mathcal{S}$, then

$$(x,y) \in \rho \Longleftrightarrow ax = ay \text{ for some } a \in \mathcal{S}$$
$$\Longrightarrow (ax, ay) \in \rho' \Longrightarrow (x,y) \in \rho' \Longrightarrow \rho \subseteq \rho'.$$

This establishes that ρ is the smallest congruence on $\mathcal{S}$. ■

This result has the following dual

Proposition 7.1.5 *Let $\mathcal{S}$ be an L-semi-commutative semigroup. Then the relation ρ^* defined on $\mathcal{S}$ by*

$$x\ \rho^*\ y \Longleftrightarrow xa = ya \ \text{ for some } a \in \mathcal{S}$$

is the smallest cancellative congruence on $\mathcal{S}$.

We now immediately deduce

Corollary 7.1.3 *Let $\mathcal{S}$ be a semi-commutative semigroup, and let ρ, ρ^* be as in Propositions 7.1.3 and 7.1.4. Then $\rho = \rho^*$.*

7.2 Semigroups of fractions

R and L-semi-commutative semigroups are well-suited to the construction of fractions. In this section, we will construct semigroups of fractions of R-semi-commutative semigroups in a similar way to that of the commutative semigroups; and we will see that most of the results in commutative semigroups of fractions still hold for L and R-semi-commutative semigroups of fractions.

Let $\mathcal{S}$ be an R-semi-commutative semigroup and K be a subsemigroup of $\mathcal{S}$. The semigroup of fractions $K^{-1}\mathcal{S}$ is constructed as follows:

Let $\sim$ be the binary relation on $\mathcal{S} \times K^1$ defined by

$$(x, a) \sim (y, b) \Longleftrightarrow cay = cbx \text{ for some } c \in K^1.$$

The next lemma shows that $\sim$ is a congruence on $\mathcal{S} \times K^1$.

Lemma 7.2.1 *The binary relation $\sim$ is a congruence on $S \times K^1$.*

Proof. Clearly $\sim$ is symmetric and reflexive. To show that $\sim$ is transitive, let

$$(x, a) \sim (y, b),\ (y, b) \sim (z, c) : x, y, z \in \mathcal{S}, a, b, c \in K^1.$$

Then, there exist $d, e \in K^1$ such that

$$day = dbx,\ ebz = ecy.$$

Hence,

$$ae^2c^2(dbx) = ae^2c^2(day) = aecda(ecy) = aecda(ebz)$$
$$= ae^2cdb(az) \Longrightarrow ae^2cdb(az) = ae^2c^2(dbx) = ae^2cdb(cx).$$

Since $a, b, c, d, e \in K^1$, $ae^2cdb \in K^1$ and thus $(x, a) \sim (z, c)$ which shows that $\sim$ is transitive. Now we show that if $(x, a) \sim (y, b)$ and $(z, c) \in \mathcal{S} \times K^1$

then $(xz, ac) \sim (yz, bc)$ and $(zx, ca) \sim (zy, cb)$. So let $(x, a) \sim (y, b)$ and $(z, c) \in \mathcal{S} \times K^1$. Then there exists $d \in K^1$ such that

$$day = dbx \Longrightarrow c^2abdz(day) = c^2abdz(dbx)$$

$$\Longrightarrow c^2abd^2z(ay) = cabd^2(cazy) = cabd^2(cbzx).$$

Similarly,

$$c^2(day)z = c^2(dbx)z \Longrightarrow cd(acyz) = cd(bcxz).$$

Hence, $\sim$ is a congruence. ∎

Let the fraction $x/a : x \in \mathcal{S}, a \in K^1$ be the equivalence class of (x, a). Then, the semigroup of fractions $K^{-1}\mathcal{S}$ is the quotient semigroup $\mathcal{S} \times K^1/\sim$. Clearly, $K^{-1}\mathcal{S}$ is an R-semi-commutative monoid in which the elements of K are units. Also, if $K \neq \emptyset$ and $x, y \in \mathcal{S},\ a, b \in K^1$, then

$$(x/a)(y/b) = (xy)/(ab)$$

and

$$x/a = y/b \Longleftrightarrow \exists\ c \in K^1\ :\ cay = cbx.$$

The mapping $\alpha : \mathcal{S} \longrightarrow K^{-1}\mathcal{S},\ x \longmapsto x/1$ is a homomorphism. Furthermore, for each $a \in K,\ (a)\alpha$ is a unit of $K^{-1}\mathcal{S}$.

Definition 7.2.1 Let $\alpha : \mathcal{S} \longrightarrow S^*,\ \phi : \mathcal{S} \longrightarrow T$ be semigroup homomorphisms. Then ϕ *factors* uniquely through α if there exists a unique homomorphism $\psi : S^* \longrightarrow T$ such that $\alpha \circ \psi = \phi$.

Proposition 7.2.1 *Let $\mathcal{S}$ be an R-semi-commutative semigroup, let $K \neq \emptyset$ be a subsemigroup of $\mathcal{S}$, and let T be an R-semi-commutative monoid. If $\alpha : \mathcal{S} \longrightarrow K^{-1}\mathcal{S}$ is the homomorphism which takes $x \longmapsto x/1$ and $\phi : \mathcal{S} \longrightarrow T$ is a homomorphism such that $(a)\phi$ is a unit of T for every $a \in K$, then ϕ factors uniquely through α.*

Proof. If $\mathcal{S} \neq \mathcal{S}^1$, we extend ϕ to $\mathcal{S}^1$ such that $(1)\phi = 1$. If $\mathcal{S} = \mathcal{S}^1$, then for $a \in K \neq \emptyset$ we get

$$\begin{aligned}
&(a)\phi = (1.a)\phi = (1)\phi\ (a)\phi \\
&\Longrightarrow (a)\phi\ ((a)\phi)^{-1} = (1)\phi\ (a)\phi\ ((a)\phi)^{-1} \\
&\Longrightarrow 1 = (1)\phi \quad (\text{as } (a)\phi \text{ is a unit of } T\ ).
\end{aligned} \tag{7.1}$$

Let $\psi : K^{-1}\mathcal{S} \longrightarrow T$ be a homomorphism such that $\alpha \circ \psi = \phi$. Let $a \in K$. Since $(a)\alpha$ and $(a)\phi$ are units and $(a)\alpha \circ \psi = (a)\phi$, we have

$$\begin{aligned}
&(a)\phi = (a)\alpha \circ \psi = (1.(a)\alpha)\psi \\
&= (1)\psi\ ((a)\alpha)\psi = (1)\psi\ (a)\phi.
\end{aligned}$$

Hence,

$$(a)\phi = (1)\psi\ (a)\phi. \tag{7.2}$$

Multiplying both sides of (7.2) by $((a)\phi)^{-1}$ from the right gives $(1)\psi = 1$. Hence, $(a/a^2) = ((a)\alpha)^{-1}$ in $K^{-1}\mathcal{S}$ and

$$(a/a^2)\psi = (((a)\alpha)^{-1})\psi = ((a)\phi)^{-1} \quad \text{in } T. \tag{7.3}$$

If $x \in \mathcal{S}$, then

$$(x/a) = (x/1)(a/a^2) = (x)\alpha((a)\alpha)^{-1}. \tag{7.4}$$

Thus,

$$\forall x \in \mathcal{S}, \forall a \in K^1; (x/a) = (x)\alpha\ ((a)\alpha)^{-1};$$

and

$$\begin{aligned}
&(x/a)\psi = ((x)\alpha\ ((a)\alpha)^{-1})\psi \quad (\text{by } (7.4)\ ) \\
&= ((x)\alpha \circ \psi)(((a)\alpha)^{-1} \circ \psi) = (x)\phi\ ((a)\phi)^{-1}
\end{aligned}$$

where $(1)\phi = 1$ (by (7.1)).

If $(x/a) = (y/b)$ in $K^{-1}\mathcal{S}$ then $cay = cbx$ for some $c \in K^1$ and thus

$$(c)\phi\ (a)\phi\ (y)\phi = (c)\phi\ (b)\phi\ (x)\phi$$

$$\begin{aligned}
&\Longrightarrow (a)\phi\ (y)\phi = (b)\phi\ (x)\phi \ \ (\text{as } (c)\phi \text{ is a unit in } T)\\
&\Longrightarrow (y)\phi = ((a)\phi)^{-1}\ (b)\phi\ (x)\phi \ \ (\text{ as } (a)\phi \text{ is a unit in } T)\\
&\Longrightarrow (y)\phi\ ((b)\phi)^{-1} = ((a)\phi)^{-1}\ (b)\phi\ (x)\phi\ ((b)\phi)^{-1}\\
&\quad = ((a)\phi)^{-1}\ (b)\phi\ ((b)\phi\ ((b)\phi)^{-1})\ (x)\phi\ ((b)\phi)^{-1}\\
&\quad = ((a)\phi)^{-1}\ (b)\phi\ ((b)\phi\ ((b)\phi)^{-2})\ (x)\phi\\
&\quad = ((a)\phi)^{-1}.1.(x)\phi = ((a)\phi)^{-1}\ (x)\phi.
\end{aligned}$$

Hence,

$$(y)\phi\ ((b)\phi)^{-1}\ (a)\phi = ((a)\phi)^{-2}(a)\ \phi\ (x)\phi\ (a)\phi = 1.(x)\phi = (x)\phi. \tag{7.5}$$

Multiplying both sides of (7.5) by $((a)\phi)^{-1}$ from the right gives

$$(y/b)\psi = (y)\phi\ ((b)\phi)^{-1} = (x)\phi\ ((a)\phi)^{-1} = (x/a)\psi.$$

This implies that in the above ψ is unique, and that a mapping $\psi : K^{-1}\mathcal{S} \longrightarrow T$ is well-defined by $(x/a)\psi = (x)\phi\ ((a)\phi)^{-1}$. Also, ψ is a homomorphism because:

$$\begin{aligned}
((x/a)(y/b))\psi = ((xy)/(ab))\psi = (xy)\phi\,((ab)\phi)^{-1} &= (x)\phi\,(y)\phi\,((b)\phi)^{-1}\,((a)\phi)^{-1}\\
&= (x)\phi\ ((a)\phi\ ((a)\phi)^{-1})\ (y)\phi\ ((b)\phi)^{-1}\ ((a)\phi)^{-1}\\
&= (x)\phi\ ((a)\phi)^{-1}\ (y)\phi\ ((b)\phi)^{-1} = (x/a)\psi\ (y/b)\psi.
\end{aligned}$$

It then follows from the definition of ψ that $\alpha \circ \psi = \phi$. ∎

The proof of the next proposition is almost the same as the proof for the commutative case as it only depends on Proposition 7.2.1.

Proposition 7.2.2 *Let $\mathcal{S}$ be an R-semi-commutative semigroup, let $K \neq \emptyset$ be a subsemigroup of $\mathcal{S}$, and let T be an R-semi-commutative monoid such that $\mathcal{S}$ is a subsemigroup of T. Then, $K^{-1}\mathcal{S}$ is isomorphic to a subsemigroup of $K^{-1}T$.*

Proof. Let $\phi : \mathcal{S} \longrightarrow K^{-1}T$ be the restriction of $\alpha : T \longrightarrow K^{-1}T$ to $\mathcal{S}$. By Proposition 7.2.1 there exists a homomorphism $\psi : K^{-1}\mathcal{S} \longrightarrow K^{-1}T$ such that $(x/a)\psi = (x)\phi((a)\phi)^{-1}$. It then follows from (7.4) that, for every x/a in $K^{-1}\mathcal{S}$, we have

$$(x/a)\psi = (x)\phi\ ((a)\phi)^{-1} = (x)\alpha\ ((a)\alpha)^{-1} = x/a \in K^{-1}T.$$

Let $x, y \in \mathcal{S},\ a, b \in K^1$. If $x/a = y/b$ in $K^{-1}T$, then $cay = cbx$ for some $c \in K^1$ and $x/a = y/b$ in $K^{-1}\mathcal{S}$; thus ψ is injective and $K^{-1}\mathcal{S} \cong (K^{-1}\mathcal{S})\psi$. ∎

Now we apply the semigroup of fractions to universal groups.

Definition 7.2.2 The *universal group* of a semigroup $\mathcal{S}$ is a group $G(\mathcal{S})$ together with a homomorphism $\gamma : \mathcal{S} \longrightarrow G(\mathcal{S})$, such that every homomorphism of $\mathcal{S}$ into a group factors uniquely through γ.

Proposition 7.2.3 *Let $\mathcal{S}$ be a non-empty R-semi-commutative semigroup, and let $\alpha : \mathcal{S} \longrightarrow \mathcal{S}^{-1}\mathcal{S}$ be defined by $(x)\alpha = x/1$. Then, $\mathcal{S}^{-1}\mathcal{S}$ with α is the universal group of $\mathcal{S}$.*

Proof. Clearly, $\mathcal{S}^{-1}\mathcal{S}$ is a group. Let G be a group, and let $\phi : \mathcal{S} \longrightarrow G$ be a homomorphism. Since G is a group, $(x)\phi$ is a unit in G for every $x \in \mathcal{S}$. It then follows from Proposition 7.2.1 that ϕ factors uniquely through α. This shows that $\mathcal{S}^{-1}\mathcal{S}$, along with α, is the universal group of $\mathcal{S}$. ∎

Proposition 7.2.4 *Let $\mathcal{S}$ be an R-semi-commutative semigroup, and let $I \neq \emptyset$ be a right ideal of $\mathcal{S}$. Then, $G(I) \cong G(\mathcal{S})$ where $G(I)$, $G(\mathcal{S})$ are the universal groups of $I, \mathcal{S}$ respectively.*

Proof. Similar to the proof for the commutative semigroups. ∎

The following two propositions are immediate.

Proposition 7.2.5 *Let $\mathcal{S}$ and T be left-cancellative R-semi-commutative semigroups. If $\mathcal{S}$ is a subsemigroup of T, then $G(\mathcal{S})$ is isomorphic to a subgroup of $G(T)$.* ■

Proposition 7.2.6 *Let $\mathcal{S}$ and T be non-empty left-cancellative R-semi-commutative semigroups. Then,*

$$G(\mathcal{S} \times T) \cong G(\mathcal{S}) \times G(T). \quad ■$$

7.3 Archimedean R-semi-commutative semigroups

In this section we generalize some of the properties of archimedean commutative semigroups to archimedean R-semi-commutative semigroups.

Definition 7.3.1 A semigroup $\mathcal{S}$ is said to be an *archimedean semigroup* if, for each $a, b \in \mathcal{S}$, there are positive integers m and n such that $a^m \in \mathcal{S}b\mathcal{S}$ and $b^n \in \mathcal{S}a\mathcal{S}$.

Proposition 7.3.1 *Let $\mathcal{S}$ be an R-semi-commutative semigroup, and let $a, b \in \mathcal{S}$. Define η as the set of pairs (a, b) in $\mathcal{S} \times \mathcal{S}$ for which*

$$(\exists m \in \mathbf{N},\ x, y \in \mathcal{S})\ a^m = xby \quad \textit{and} \quad (\exists n \in \mathbf{N},\ z, w \in \mathcal{S})\ b^n = zaw.$$

Then, η is a congruence on $\mathcal{S}$; and $\mathcal{S}/\eta$ is a semilattice.

Proof. Clearly, η is symmetric and reflexive. To show that η is transitive, let $(a, b), (b, c) \in \eta$. Then there exist $m, n, r, k \in \mathbf{N}$, $x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4 \in \mathcal{S}$ such that

$$a^m = x_1by_1,\ b^n = x_2ay_2,\ b^r = x_3cy_3,\ c^k = x_4by_4.$$

Thus,

$$a^{mr} = (x_1by_1)^r = x_1^r b^r y_1^r = x_1^r(x_3cy_3)y_1^r, \tag{7.6}$$

$$c^{kn} = (x_4by_4)^n = x_4^n b^n y_4^n = x_4^n(x_2ay_2)y_4^n. \tag{7.7}$$

It then follows from (7.6) and (7.7) that $(a, c) \in \eta$ and η is transitive.

Let $(a, b) \in \eta$, and let $c, d \in \mathcal{S}$. (We want to show that $(cad, cbd) \in \eta$.) Then

$$(\exists m \in \mathbf{N},\, x_1, y_1 \in \mathcal{S})\ a^m = x_1by_1 \ \text{ and } \ (\exists n \in \mathbf{N},\, x_2, y_2 \in \mathcal{S})\ b^n = x_2ay_2.$$

Thus,

$$\begin{aligned}(cad)^{m+1} &= c^{m+1}a^{m+1}d^{m+1} = c^{m+1}ad^{m+1}a^m \\ &= c^{m+1}ad^{m+1}(x_1by_1) = c^m ad^m(x_1cbdy_1).\end{aligned}$$

Hence,

$$(cad)^{m+1} = c^m ad^m x_1(cbd)y_1. \tag{7.8}$$

Similarly,

$$\begin{aligned}(cbd)^{n+1} &= c^{n+1}b^{n+1}d^{n+1} = c^{n+1}bd^{n+1}b^n \\ &= c^{n+1}bdn + 1(x_2ay_2) = c^n bd^m(x_2cady_2),\end{aligned}$$

and

$$(cbd)^{n+1} = c^n bd^n x_2(cad)y_2. \tag{7.9}$$

It then follows from (7.8) and (7.9) that $(cad, cbd) \in \eta$. Hence, η is a congruence on $\mathcal{S}$.

It is now immediate that $\mathcal{S}/\eta$ is a band. To show that $\mathcal{S}/\ \eta$ is commutative, note that

$$\forall\ a, b \in \mathcal{S}, (ab)^2 = a^2b^2 = a(ba)b \text{ and } (ba)^2 = b(ab)a.$$

Hence, $[ab]_\eta = [ba]_\eta$ and $[a]_\eta[b]_\eta = [b]_\eta[a]_\eta$. ∎

Corollary 7.3.1 *Each η-class of $\mathcal{S}$ is an archimedean subsemigroup of $\mathcal{S}$ and hence every R-semi-commutative semigroup can be expressed as a semilattice of archimedean R-semi-commutative semigroups.*

Proof. Assume that $x \ \eta \ y$. Then

$$(\exists x_1, x_2 \in \mathcal{S},\ m \in \mathbf{N}),\ x^m = x_1 y x_2 \ \text{ and } \ (\exists y_1, y_2 \in \mathcal{S},\ n \in \mathbf{N}),\ y^n = y_1 x y_2.$$

Let ℓ be the least common multiple of m, n. Then, $\ell = mr = nt$ for some $r, t \in \mathbf{N}$ and

$$\begin{aligned}(xy)^\ell = x^\ell y^\ell = (x^m)^r (y^n)^t &= (x_1^r y^r x_2^r)(y_1^t x^t y_2^t) \\ &= x_1^r y^r x_2^r y_1^t \underbrace{(x)}\, x^{t-1} y_2^t. \end{aligned} \tag{7.10}$$

Also,

$$x^{m+2} = x(x^m)x = x(x_1 y x_2)x = x x_1 \underbrace{(xy)}\, x_2. \tag{7.11}$$

It then follows from (7.10) and (7.11) that $x \ \eta \ xy$ and thus the η-class containing x, y is closed and thus is a subsemigroup of $\mathcal{S}$. This shows that each η-class in $\mathcal{S}$ is an archimedean subsemigroup of $\mathcal{S}$. ∎

The last statement of Corollary 7.3.1 is special case of the following Theorem.

Theorem 7.3.1 ([50] *Every exponential semigroup is a semilattice of archimedean semigroups.*

Next, we show that the definition of the smallest cancellative congruence on an archimedean commutative semigroup defines the smallest cancellative congruence on an R-semi-commutative semigroup.

Lemma 7.3.1 *Let $\mathcal{S}$ be an R-semi-commutative semigroup and let $a \in \mathcal{S}$. Then the relation ξ on $\mathcal{S}$ defined by*

$$x \ \xi \ y \iff a^n x = a^n y \quad \textit{for some } n > 0$$

is a congruence on $\mathcal{S}$.

Proof. Clearly, ξ is symmetric and reflexive. Also, if $a^n x = a^n y$ and $a^k y = a^k z$, then

$$a^{n+k}x = a^k a^n x = a^k a^n y = a^n a^k y = a^{n+k}z.$$

Hence, ξ is transitive and thus an equivalence relation. To show that ξ is a congruence, let $(x,y) \in \xi,\ c,d \in \mathcal{S}$. Then, $a^n x = a^n y$ for some $n > 0$ and

$$a^{n+1}(cxd) = ac\underbrace{(a^n x)}\,d = ac\underbrace{(a^n y)}\,d = a^{n+1}(cyd). \tag{7.12}$$

Hence, (7.12) shows that the equivalence relation ξ is a congruence. ■

The next proposition shows that ξ is the smallest cancellative congruence on an archimedean R-semi-commutative semigroup.

Proposition 7.3.2 *Let $\mathcal{S}$ be an archimedean R-semi-commutative semigroup, and let ξ and a be as in Lemma 7.3.1. Then, ξ is the smallest cancellative congruence on $\mathcal{S}$. Furthermore, $\mathcal{S}/\xi$ is cancellative.*

Proof. By Lemma 7.3.1 ξ is a congruence on $\mathcal{S}$. Assume that $a^n xz = a^n yz$ for some $x, y, z \in \mathcal{S},\ n \in \mathbf{N}$. Since $\mathcal{S}$ is archimedean, there exist $s, t \in \mathcal{S}$ such that $a^m = szt$ for some $m \in \mathbf{N}$. Thus,

$$\begin{aligned} a^{2m+n}x &= \underbrace{(a^m)}\,a^n x\,\underbrace{(a^m)} = \underbrace{(szt)}\,a^n x\,\underbrace{(szt)} = s^2zt^2a^n xz \\ &= s^2zt^2a^n yz = (szt)^2a^n y = (a^m)^2a^n y = a^{2m+n}y. \end{aligned}$$

This shows that $x\ \xi\ y$ and ξ is right-cancellative. Similarly, if $zx\ \xi\ zy$ then

$$\begin{aligned} a^{2m+2n}x = a^{2n}\,\underbrace{(a^m)}\,x\,\underbrace{(a^m)} &= a^{2n}\,\underbrace{(szt)}\,x\,\underbrace{(szt)} = a^n s^2zt^2a^n zx \\ &= s^2zt^2a^n zy = a^n(s^2z^2t^2)a^n y \\ &= a^n(szt)^2 = a^n a^{2m}y = a^{2m+n}y. \end{aligned}$$

Hence ξ is also left-cancellative and thus cancellative.

If ξ' is a cancellative congruence on $\mathcal{S}$, then

$$x \;\xi\; y \Longleftrightarrow a^n x = a^n y : n > 0 \Longrightarrow a^n x \;\xi'\; a^n y$$

$$\Longrightarrow x \;\xi'\; y \quad (\text{as } \xi' \text{ is cancellative}) \Longrightarrow \xi \subseteq \xi'.$$

This establishes that ξ is the smallest cancellative congruence on $\mathcal{S}$. It is now immediate that $\mathcal{S}/\xi$ is cancellative. ■

Definition 7.3.2 A cancellative archimedean semigroup $\mathcal{S}$ is called N-*semigroup* if $\mathcal{S}$ contains no idempotent.

We need the next fact to show that $\mathcal{S}/\xi$ is an N-semigroup whenever $\mathcal{S}$ has no idempotent.

Lemma 7.3.2 *Let $\mathcal{S}$ be an archimedean R-semi-commutative semigroup with no idempotent. Then $yx \neq x$ for all $x, y \in \mathcal{S}$.*

Proof. Assume that $yx = x$ for some $x, y \in \mathcal{S}$. Then $y^n x = x$ for all $n > 0$. Also, $(x)y = (yx)y = y^2 x = x$. Thus $xy^n = x$ for all $n > 0$. Since $\mathcal{S}$ is archimedean, we have $y^k = sxt$ for some $k \in \mathbf{N}$ and $s, t \in \mathcal{S}$; and

$$x = xy^k = x(sxt) = x^2 st \Longrightarrow x^2 \in \mathcal{R}_x; \tag{7.13}$$

$$x = y^{2k}x = y^k x y^k = sxt(x)sxt = s^2xt^2x^2 \Longrightarrow x^2 \in \mathcal{L}_x. \tag{7.14}$$

It then follows from (7.13) and (7.14) that $x^2 \in H_x$. Hence, $H_x \cap H_x^2 \neq \emptyset$. It then follows by Theorem 1.5.1 that H_x is a subgroup of $\mathcal{S}$ and $\mathcal{S}$ contains an idempotent, which contradicts the hypothesis. ■

For L-semi-commutative semigroups we have:

Lemma 7.3.3 *If $\mathcal{S}$ is an archimedean L-semi-commutative semigroup with no idempotent, then $xy \neq x$ for all $x, y \in \mathcal{S}$.*

Proof. The proof of this lemma is similar to the proof of Lemma 7.3.2. ■

Proposition 7.3.3 *Let $\mathcal{S}$ be an R-semi-commutative semigroup without an idempotent, and let ξ be as in Lemma 7.3.1. Then $\mathcal{S}/\xi$ is an N-semigroup.*

Proof. By Proposition 7.3.2 $\mathcal{S}/\xi$ is cancellative. Also, $\mathcal{S}/\xi$ is archimedean as $\mathcal{S}$ is. If $\mathcal{S}/\xi$ contains an idempotent, then $a^n x = a^n x^2$ for some $n \in \mathbf{N}$ and $x \in \mathcal{S}$. Hence,

$$x^2 a^n = x a^n x = x a^n x^2 = x^3 a^n = x(x^2 a^n). \tag{7.15}$$

By Lemma 7.3.2 the relation in (7.15) cannot happen when $\mathcal{S}$ has no idempotent. ■

All the above mentioned results of R-semi-commutative semigroups can be easily modified for L-semi-commutative semigroups.

We now let $\mathcal{S}$ be a semi-commutative semigroup.

Lemma 7.3.4 *Let $\mathcal{S}$ be a semi-commutative semigroup, and let $e \in E(\mathcal{S})$. Then, $ex = xe$ for any $x \in \mathcal{S}$.*

Proof. $ex = e^2 x = exe = xe^2 = xe.$ ■

Lemma 7.3.5 *An archimedean semi-commutative semigroup $\mathcal{S}$ contains at most one idempotent.*

Proof. Assume that $e, f \in E(\mathcal{S})$. Since $\mathcal{S}$ is archimedean, $e = xfy,\ f = zew$ for some $x, y, z, w \in \mathcal{S}$. Thus

$$\begin{aligned} ef &= e(zew) = zew = \underbrace{f} = xey = xeye \\ &= fe = f(xfy) = xfy = \underbrace{e}. \end{aligned}$$ ■

Definition 7.3.3 A semigroup $\mathcal{S}$ is called *nilsemigroup* if $\mathcal{S}$ contains zero element and for each $x \in \mathcal{S}$, there exists $n \in \mathbf{N}$ such that $x^n = 0$.

Definition 7.3.4 Let I be an ideal of a semigroup $\mathcal{S}$. Then the relation ϱ defined by

$$a \ \varrho \ b \Longleftrightarrow a = b \text{ or } a, b \in I$$

is called a *Rees congruence.* The quotient semigroup $\mathcal{S}/I$ $(= \mathcal{S}/\varrho)$ is called the *Rees quotient of $\mathcal{S}$ by I.*

Definition 7.3.5 Let $\mathcal{S}$ and T be non-empty disjoint semigroups with T containing a zero element. An *ideal extension* of $\mathcal{S}$ by T is a semigroup $\mathcal{S}^*$ such that $\mathcal{S}$ is an ideal of $\mathcal{S}^*$ and the Rees quotient $\mathcal{S}^*/\mathcal{S} = T$.

Proposition 7.3.4 *Let $\mathcal{S}$ be an archimedean semi-commutative semigroup. Then $\mathcal{S}$ contains an idempotent if and only if $\mathcal{S}$ is either a group or an ideal extension of a group by a nilsemigroup.*

Proof. Let $\mathcal{S}$ be an archimedean semi-commutative semigroup, and let $e \in E(\mathcal{S})$. Then,

$$e = xay, \ a^n = zew : n > 0, \ x, y, z, w \in \mathcal{S}.$$

Thus,

$$\mathcal{S}^1 ea = \mathcal{S}^1 eae \subseteq \mathcal{S}^1 e, \tag{7.16}$$

$$\begin{aligned} \mathcal{S}^1 e = \mathcal{S}^1 xay = \mathcal{S}^1 x^2 a^2 y^2 = \mathcal{S}^1 x^2 ay^2 a \\ = \mathcal{S}^1 xy(xay)a \subseteq \mathcal{S}^1 (xay)a = \mathcal{S}^1 ea. \end{aligned} \tag{7.17}$$

It then follows from (7.17) and (7.18) that $\mathcal{S}^1 e = \mathcal{S}^1 ea$ and $ea \in \mathcal{L}_e$. Similarly, $e\mathcal{S}^1 = ea\mathcal{S}^1$ and $ea \in \mathcal{R}_e$. This shows that $ea \in H_e$. Since a is arbitrary, we have

$$\mathcal{S}e = H_e = e\mathcal{S} \quad (\text{ by Lemma 7.3.5 }).$$

Hence, the subgroup H_e of $\mathcal{S}$ is an ideal. Also, every $a \in \mathcal{S}$ has a power in H_e because $a^n = zew$ for some $n > 0,\ z, w \in \mathcal{S}$ and

$$a^{2n} = (zew)^2 = z^2e^2w^2 = e^2z^2w^2 = z^2w^2e \in H_e.$$

Hence, if $\mathcal{S} \neq H_e$ then $\mathcal{S}/H_e$ is a nilsemigroup.

Conversely, let $\mathcal{S}$ be an ideal extension of a group G by a nilsemigroup N $(N = \mathcal{S}/G)$. Then the identity element e of G is an idempotent of $\mathcal{S}$. Let $a \in \mathcal{S}$. If $a \notin G$, then $a^n = 0$ holds in $N = \mathcal{S}/G$ for some n, so that $a^n \in G$ and

$$a^n = a^ne = a^ne^2 = a^n(e)e \text{ and } e = a^{n-1}(a)(a^n)^{-1}.$$

Hence, if $a, b \in \mathcal{S},\ a \notin G, b \in G$ then

$$a^n = a^ne = a^n(b)b^{-1} \text{ and } b = be = ba^{n-1}(a)(a^n)^{-1}.$$

While if $a, b \notin G$ then $a^n = 0,\ b^m = 0$ hold in $N = \mathcal{S}/G$ for some n, m, so that $a^n, b^m \in G$ and

$$a^n = b^{m-1}(b)(b^m)^{-1} \text{ and } b^m = a^{n-1}a(a^n)^{-1}.$$

Thus $\mathcal{S}$ is archimedean. Also, if $\mathcal{S}$ is a group then

$$(\forall x, y \in \mathcal{S})xyx = x^2y \iff x^{-1}xyx = x^{-1}x^2y \iff yx = xy.$$

Hence, $\mathcal{S}$ is abelian, and abelian semigroups are archimedean. ∎

7.4 Conclusion and open problems

The importance of this thesis is not only in the results obtained but also in the techniques used to obtain these results.

The techniques used in Chapters 4, 5 and 6 revealed the structure of the semigroups defined by the presentations considered. Those techniques can be applied to various types of semigroup presentations and may reveal further interesting results.

One of the main results of Chapter 6 was that the Schützenberger group of a non-regular $\mathcal{H}$-class in a commutative, R-semi-commutative or L semi-commutative semigroup is a homomorphic image of a regular $\mathcal{H}$-class. Further investigation may lead to a larger class of semigroups satisfying this result and may reveal under what conditions they are isomorphic as we did for the above cases.

We have seen that the classes of semigroups described in Chapter 7, and commutative semigroups have many properties in common. Further study of these classes may extend other results of commutative semigroups to these classes.

The author intends to carry on investigating other semigroup presentations in the light of the techniques and results of Chapters 3, 4 and 5; and working on the following open problems:

Problem 7.4.1 Let $\mathcal{S}$ be a semigroup such that there exists an integer n and $x^n yx = x^{n+1}y$ for all $x, y \in \mathcal{S}$. We call $\mathcal{S}$ an nR-semi-commutative semigroup. For which $n \geq 2$ is the Schützenberger group of any non-regular $\mathcal{H}$-class in an nR-semi-commutative semigroup a homomorphic image of a regular $\mathcal{H}$-class? An nL-semi-commutative semigroup is defined in the same way.

Problem 7.4.2 Let $\mathcal{S}^m$ be an exponential semigroup. Is any Schützenberger group of a non-regular $\mathcal{H}$-class in $\mathcal{S}$ a homomorphic image of a regular $\mathcal{H}$-class?

Problem 7.4.3 What is the biggest class of semigroups in which the Schützenberger group of any non-regular $\mathcal{H}$-class is a homomorphic image of a regular $\mathcal{H}$-class?

Problem 7.4.4 Let H^* be a regular $\mathcal{H}$-class in a semigroup $\mathcal{S}$, and let H be a non-regular $\mathcal{H}$-class such that the Schützenberger group of H is a homomorphic image of H^*. Under what conditions is the Schützenberger group of H isomorphic to H^*?

Problem 7.4.5 Is a finitely generated semi-commutative semigroup finitely presented?

Problem 7.4.6 Is a finitely generated (L) R-semi-commutative semigroup finitely presented?

Remark 7.4.1 Problem 7.4.5 has been solved by Dr. Geoff Smith.

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